A chaotic approach to dynamic asset pricing theory

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A dissertation submitted to the University of London in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics *dedicated to Galini*

I tell you: you must have chaos in you
to give birth to a dancing star

Friedrich Nietzsche (1844-1900)

Thus Spoke Zarathustra, A Book for All and None

Acknowledgments

It has been a fascinating three–year journey, and I feel fortunate to have met various individuals during my trajectory as a research student at King's College London, characters that have irrevocably shaped my taste and ability in mathematics. First, and foremost, I would like to thank Lane Hughston for his constant encouragement and support during this project. Most of all, I would like to thank him for teaching me how to think outside the box, and for challenging me all the time to be original. His original style, his collection of anecdotes attached to each idea, and of course the ideas themselves, have been a source of inspiration. Had I to decide again, there is no doubt in my mind, all my choices would have been the same.

Still at King's College, I would like to thank Mihail Zervos for many very helpful discussions and suggestions during the last three years, and Martijn Pistorius for valuable comments during the later stages of this work. Many thanks also to all members of staff, especially from the Financial Mathematics group. It is fair to say that I felt like a member of a big family from the very beginning and this played a crucial role in the development of this work. To my fellow PhD students I owe a big thank you not only for countless discussions about mathematics, but also for being an integral part of my life for the last three years. In particular, my special thanks to my friends Amal Merhi, Andrew Jack, and Andrea Macrina. I will never forget the 'Williams sessions' during the first year, and all the fun we had learning probability.

Outside King's, it is a pleasure to mention Henrik Rasmussen and thank him for all his suggestions and support during the years, and for putting me in the right track. I would also like to thank Dorje Brody for many discussions concerning my work, especially during the last year. Many friends have managed to put up with me while I was putting up with mathematics. I thank them all and in particular Stefano Galiani, Nektaria Karakatsani, Marios Kiriakou, Eleni Lianta, Greg Pavliotis, Ioannis Pilidis, George Poursanidis, Veronika Systik, Ioannis Theodoridis, Maria Tzortziou and Antonis Zagaris.

For financial support, I would like to thank the Department of Mathematics at King's College London and the Empirikio Foundation, Athens.

My family has been right there with me from the very beginning, and nothing would have ever been possible without their constant encouragement, love and support. My parents Ioannis and Galini, my brothers Kostas and Agellos, and my sister Sofia, have always been supportive of my choices and they have been the driving spirit behind all my work. If it were not for you, I would not have started this.

Avraam Rafailidis

December 2004

Abstract

We provide a new framework for modelling interest rates and the dynamics of asset prices. We consider the general family of arbitrage–free positive interest rate models, valid on all time horizons. The conditional variance representation for the pricing kernel process provides a powerful framework in which we develop the theory of discount bond evolution, when the market is driven by a Brownian motion of one or more dimensions. We show that there is a mapping between all positive arbitrage–free interest rate models and the space of square–integrable Wiener functionals. The Wiener chaos expansion technique is then used to formulate a systematic analysis of the structure and classification of interest rate models. The specification of a first–chaos model is equivalent to the specification of an admissible initial yield curve. A comprehensive development of the second–chaos interest rate theory is presented in the case of a single Brownian factor, and we show that there is a natural methodology for calibrating the model to at–the–money–forward caplet prices. The factorisable second–chaos models are particularly tractable, and lead to closed–form expressions for various types of interest rate derivatives. The methodology is powerful enough to apply in a general foreign exchange setting, for which each currency admits an associated family of discount bonds. The same arguments hold for general asset dynamics, and we provide a generic way for creating models for trajectories of assets of limited liability. Examples include the usual geometric Brownian motion model as well as other more sophisticated models. In the case of a combination of two chaos expansions we are lead to a potentially powerful modelling alternative for the pricing of derivative products. Explicit results are given for simple European options and we discuss generalisations for more exotic derivatives.

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Chapter 1

Introduction

1.1 Initial remarks

The mathematical theory of finance has evolved over the last thirty–five years into a major scientific discipline. It is now considered an integral part of modern applied mathematics, and respected as such. What we call mathematical finance is the science of the financial markets. The market itself is a massive industry spread over the world comprising substantial volumes of 'assets' that are being bought and sold at virtually any place in the world and at any time. It would not be an exaggeration to assign the phrase remarkable to the existence and functioning of this system. As in any other discipline of science one can distinguish the theoretical and the experimental part of finance. One might say that the experimental part is the market itself, whereas mathematical finance is the theoretical part.

Despite many similarities of mathematical finance to natural sciences, it is appropriate also to acknowledge here the major difference, namely the 'human' interface at which the random market events occur. For example, in physics one is trying to understand, explain, and predict the way nature works. Therefore one is lead to experimental physics, which is a way to acquire feedback from nature, and to theoretical physics, which is effectively the construction of mathematical models in order to explain these natural phenomena. In finance the underlying randomness does not usually come from nature, but from humans instead. It is the behaviour of the market that one is trying to understand and predict, and the market is a human development. We could argue therefore that it is people who make the rules in finance, and the main difference with traditional science is that these rules change with time. In natural sciences a fundamental assumption is that laws are basically the same everywhere in the universe, and over all time. This is the principle of homogeneity in space and time. Such a fundamental principle does not exist *a priori* in finance. The way asset prices evolve heavily depends on the behaviour of market players, whose psychology and trading habits change over time. A point of view which is inexorably connected with modern physics is the fact that humans after all are part of nature, and therefore it is again nature that one is trying to model here. Nevertheless it is useful to maintain a distinction between the natural sciences and economics, and to consider the intersection of the two from the mathematical modelling point of view. In such a case, one is led to believe that the lack of fundamental assumptions such as homogeneity, as well as the rather minimal amount of knowledge on brain functioning, add to the magnitude of difficulty in any attempt to be 'precise' and 'rational', in other words, to be scientific, when modelling the market. It is therefore proper to assume that more than the traditional techniques from natural sciences are needed. Finance is the part of this development that corresponds to science. It represents the part of the subject that we can develop with traditional mathematical techniques and ideas.

As mentioned already, mathematical finance is the theoretical part of this science, and that is the topic of the present study. However, before we embark on this investigation it would be fruitful to discuss briefly the other substantial element of the science of finance, namely the experimental part, or *applied finance*. The motivation for theoretical efforts often come from practical problems. Especially in this area of intellectual effort, there would indeed be little reason to justify mathematical work if it were not for the existence of the global financial system. A major development in the field was the introduction of what we now call derivative securities. It has been long realised that positions, such as stocks, bonds, currencies, and commodities can be hedged in a sufficient way by the use of other assets whose value is contingent on those. Moreover, one can speculate and create a large amount of wealth, or indeed lose it. Options, futures, swaps and the like are contracts that are familiar now to everyone with a basic knowledge or experience of financial markets. Vanillas, Americans, and exotics are just a few words that characterise the complexity and sophistication of these products; the financial jargon, matching with the mentality and imagination of traders and other practitioners, is unlimited in scope.

Let us now turn to a final point that should be mentioned. Our intention is to model the market. What is indeed the correct way to do it? What is the best method to 'predict' something that is random, and what kind of mathematics should we use to create models and to solve practical problems? The view we take here, and we do not spend time justifying it, is that all mathematics is good mathematics. Nevertheless, we cannot ignore the natural randomness associated with the market, the unpredictability which is essentially the reason why the system exists. A deterministic universe could work in principle; a deterministic market would collapse in no time. Experimental observations confirm this: not everyone is rich. Based on this, we emphasise probability theory as the technical tool to begin with. That is to say, modern probability and stochastic analysis as founded during the last century by Kolmogorov, Doob, Lévy, and others. One usually sets the theoretical background by defining a probability triple $(\Omega, \mathcal{F}, P)$ which serves as the foundation of modelling considerations. On the other hand, finance is a subject in which the practicalities appear much earlier than in other fields, and no matter what the level of abstraction one is using for creating models, very quickly one is faced with the problem of generating numbers. Typically, one expects to face the problem of solving partial differential equations—another mathematical area highly applicable to financial modelling—and it is almost always the case that everything boils down to numerical calculations. We cannot emphasise enough the 'scalar' nature of asset pricing theory: all prices are numbers. The net result is a diverse spectrum of scientists and economists working in the field, and indeed even between mathematicians, few would (seriously) argue that they are familiar with all the different techniques and approaches. Therefore, we reach the point of decision from the outset: a monograph can either touch all related areas with the obvious drawback of not specialising, or specialise into a particular subfield of a specific area. Because this is a research thesis, it is inevitable that we shall follow the latter approach.

1.2 Outline of the thesis

In this thesis we concentrate on the modelling part of modern finance. By no means does this imply that we are not interested in numerical modelling, or that we do not consider numerical issues to be important.

Having made this point clear, we present now a brief outline of the work that follows. We consider the classical problem of option pricing in financial markets. The products we would like to price and hedge are virtually all derivatives, vanillas and exotics, liquids and over–the–counter. However, we do not begin by addressing this most popular problem in financial modelling. The important issue that needs to be clarified here is that *we consider a unified approach to financial modelling, having as a terminal goal to classify systematically all existing models in a natural and intuitive framework, and equally importantly, to be able, under this framework, to create new models.* Therefore, we do not solve an existing problem in the literature, but provide a new framework for asset–pricing theory. It is only after having constructed this machinery that we are addressing option pricing problems. This approach is in a sense based on the belief that a 'good' model is not one that will quickly provide us with the much–wanted numbers, but one that is intuitive and is based on reasonable assumptions, and eventually provides prices as well. This is not to repudiate the validity of existing models; the point here is that one should try to increase the sophistication of models, and not only the practical uses. This is a case where we really cannot afford to be minimalistic. We want the predictions for derivatives prices and hedges, but we want these to be based on modelling assumptions that are as general and intuitive as possible.

In this thesis we provide the basis for a theoretical framework that is big enough to generate a market for a wide range of assets that is arbitrage–free, and supports positive interest rates. We do not assume the usual no–arbitrage dynamics for assets, nor do we take for granted other standard results, such as changes of measure and risk neutrality. Instead, these will come as consequences of our axiomatic framework. To commence this short description of what is to follow, we start by considering the modelling of interest rates. Interest rates are arguably the 'elementary particles' of modern finance, because they connect different points in time; i.e., they are the tools with which we can give meaning to the time value of money. One should not forget that time is the dimension that is crucial in financial modelling: static markets make no sense. The idea is to create a space in which discount bonds and interest rates exist, rates are guaranteed to be positive, and every point of this space is a specific interest rate model. Moreover, we want these models to be of increasing sophistication; by increasing the mathematical complexity we should have a more 'advanced' model. At the same time we would like to consider other assets as well. Stocks, indices, and the like, should also be incorporated in this picture, and they are. We discover that there is a natural extension to general assets of limited liability, and we are able to give examples that account for different dynamics. The case of foreign exchange dynamics is also possible; in fact, it is through the language of domestic and 'foreign' economies that we are led to asset pricing theory. As mentioned earlier, known models should be part of the picture. Indeed, for example, a simple application of the present framework accounts for the heavily used geometric Brownian motion scenario for stocks; we show this in section 5.8. From the modelling point of view, the crucial observation we are going to make is the conditional variance representation for the pricing kernel process. The approach is based on the so–called pricing kernel method (Constantinides 1992). The view we take is that the modelling of default–free discount bonds is sufficient for an interest rate model, essentially equivalent to the HJM approach (Heath, Jarrow and Morton 1992), and the expression of bonds in terms of numeraires is of central importance. The main mathematical tool that we use, on the other hand, is a standard result in stochastic analysis, namely the Wiener chaos expansion. Putting these ingredients together, we are able to construct a space of models that are positive, arbitrage free, and with various degrees of freedom attached to each one of them. Moreover, we do not make any assumption of market completeness, and from this perspective, the framework proposed here is more general than the HJM theory. Within this term structure environment, we then present specific models of increasing mathematical complexity, and examine their properties. We then generalise to foreign exchange and general asset dynamics, and again, provide examples of specific models. In a little more detail, the rest of the present work proceeds as follows.

Chapter 2 is a short introduction of what is called the 'standard' model in finance, i.e. the use of Brownian motion as the only source of randomness. There is a large literature on this, and many models have evolved during the years; however, it is fair to say that the Brownian motion consideration, in combination with the work by Harrison and Kreps (1979) and Harrison and Pliska (1981), is what has given the community the basis on which to work. The idea of changes of numeraires is indispensable in many applications, especially when the money market account is used as such. What is not so well known, but lies at the core of the present work, is the role of the 'natural numeraire' asset and the consequences by its use. In essence this technique, from a modelling point of view, provides an alternative paradigm under which a change to some equivalent martingale measure is no longer necessary in order to derive the price–forecasting relation for assets and derivatives. We also give an overview of existing interest rate models, emphasising their similarities and differences. This chapter, a bibliographic review, is here for completeness and quick reference.

The core of the thesis lies in the third chapter. Here we present the axiomatic framework, and show how results that are *assumed* elsewhere in the literature, here come as *consequences* of our axioms. The conditional variance representation for the pricing kernel process is introduced, and we make a connection to elements of the Hilbert space $L^2(\Omega, \mathcal{F}, P)$ and interest rate models. The key ingredient is what we will call the 'generator' of an interest rate model, namely a square–integrable random variable on the Wiener space. This random variable is the terminal value of a square–integrable martingale, and it is viewed as exogenously specified. The main result of this analysis is that by the specification of an abstract entity such as a random variable we can construct an interest rate model. Still, however, to be able to build on models that are tractable, we need to take these considerations further. This we do by the use of the Wiener chaos expansion for this generator. We introduce this useful idea in section 3.10, where we also clarify that now the deterministic functions associated with the expansion serve as inputs to the model. We conclude this chapter with a review of the Wiener chaos technique.

The fourth chapter continues this investigation in terms of more practical issues, such as option pricing. We systematically build some models, and show how to derive explicit formulae for various interest rate derivatives. The first chaos theory accounts for a deterministic interest rate model that enables us to calibrate the initial term structure. The second chaos theory is the simplest model that introduces stochasticity, and the state variables here are Gaussian martingales. The factorisable second chaos model provides explicit results for various types of interest rate derivatives, which we show in sections 4.6 and 4.7. Another feature of this chapter is a recursive relationship, enabling us to provide explicit results for the pricing kernel and therefore for the discount bonds for a chaos model of any order. This is of practical importance since one avoids calculations in each case, it also suggests that there is a hidden structure in all chaos models, possibly related to Hermite polynomials. We conclude by presenting what are called coherent and incoherent models (cf. Brody and Hughston, 2004). This family of models can be viewed as a subspace of the chaotic framework, with the possibility of this subspace to cover the whole chaotic Hilbert space in the case of an incoherent model.

Chapter 5 is devoted to general asset dynamics. The main result here is that the ideas of previous chapters can be extended for an 'international' economy that supports many currencies. The analysis in this chapter relies on the generic formula (5.13) for assets of limited liability. Based on this, we generate a number of examples, and provide results for option pricing. This includes the usual geometric Brownian motion, and a number of new models of increasing complexity.

In chapter 6 we generalise the applications of our framework to two–factor models. In particular we present the second chaos two–factor theory and discuss implications to option pricing and asset dynamics with stochastic volatility.

A word on notation

Notation quickly becomes complex in term structure modelling and asset price dynamics. The default arrangement will be that each chapter is self contained, as far as notation is concerned. This is to avoid potential problems in a few cases where we use similar symbols for different entities in different chapters.

Chapter 2

The standard model

2.1 The standard model for asset price dynamics

Perhaps the most fundamental issue in financial modelling is the source of randomness that one is obliged to consider. The complexity of this decision is crucial in almost all the practical problems that arise. The specification of a probability space is normally the starting point in financial engineering, leading to the consideration of stochastic processes, filtrations, and the general probabilistic framework of modern mathematical finance (Karatzas and Shreve 2001). What is widely considered to be the 'standard' model in this framework is the consideration of models based on an n–dimensional Wiener process as the driving source of randomness in the market. The unfolding of random economic events is then modelled by the augmented natural filtration generated by this process. The no–arbitrage idea, which we introduce later, is fundamental in this framework. Whether this is a realistic situation or not is a different issue; however, from the modelling point of view it is essential. On the other hand, one also has to address the issue of completeness. The complete market hypothesis is a restrictive one, and to achieve completeness various conditions are usually imposed on the underlying market (see, e.g., Bensoussan 1997) and on the derivatives payoff. When markets are complete, pricing is achieved through hedging; the two are indistinguishable since a replicating portfolio is guaranteed to exist for every contingent claim.

Nevertheless, this is only a specific case of the general (incomplete) market environment. In incomplete markets perfect hedging is not possible; however pricing can be achieved through a 'pricing kernel' approach. This method is essentially the one we consider in this thesis and will be developed in later chapters. In a no–arbitrage economy the specification of a pricing kernel for derivatives implies a unique system of prices. The pricing kernel is in principle determined by economic arguments (though a general argument among these lines is still lacking). This includes some general treatment of market agent preferences. This on the other hand is linked with utility maximisation (see e.g., Davis 1998). In the rest of this section we provide a brief overview of well known results in this setting.

Let us consider a frictionless market that consists of $m + 1$ assets that are traded continuously. We thus apply here two basic assumptions: no friction and continuous trading. This is going to be the case for the rest of this thesis. We consider one of these assets to be the unit initialised *money market account*, whose price trajectory evolves according to the differential equation

$$dB_t = r_t B_t dt, \qquad B_0 = 1. \tag{2.1}$$

We use here r_t for the short rate at time t. One readily derives the solution to be

$$B_t = \exp\left(\int_0^t r_s ds\right). \tag{2.2}$$

Here we note that whether this asset evolves deterministically or not depends on the process $\{r_t\}$; that is to say, if the short rate process is stochastic then so is the money market account. The remaining assets are 'risky' and their prices evolve according to the system of equations

$$dS_t^i = S_t^i \mu_t^i dt + S_t^i \sum_{\alpha=1}^{n} \sigma_t^{i\alpha} dW_t^\alpha, \qquad S_0^i > 0, \qquad i = 1, \dots, m. \tag{2.3}$$

This system can be written in vector form once we consider the volatility matrix process $\{\sigma_t^{i\alpha}\}$. If we take the ith row of this matrix process, and denote it as $\{\sigma_t^i\} = \{\sigma^{i1}, \cdots, \sigma^{in}\}$, then we have:

$$dS_t^i = S_t^i \mu_t^i dt + S_t^i \sigma_t^i dW_t, \qquad S_0^i > 0, \qquad i = 1, \ldots, m, \tag{2.4}$$

where in the equation above there is an implied inner product between the vectors $\{\sigma_t^i\}$ and the vector $dW = \{dW^1, dW^2, \cdots, dW^n\}$. Here we have additionally considered the vector process $\{\mu_t\} = \{\mu^1, \cdots, \mu^m\}$ which serves as the drift. The dividend rate processes will be denoted $\{\Delta_t\} = \{\Delta^1, \cdots \Delta^m\}$. That is to say, we allow for one share of the risky asset i to pay dividends at the rate Δ_t^i. We assume that these processes, as well as the volatility matrix $\{\sigma_t\} = \{\sigma_t^{i\alpha}\}$ and the short rate $\{r_t\}$, are $\{\mathcal{F}_t\}$–adapted and bounded. In this way we assure that the integrals under consideration exist. By use of standard Itô calculus one can show that the solution to the system of equations (2.4) is

$$S_t^i = S_0^i \exp\left(\int_0^t \left[\mu_s^i - \frac{1}{2}|\sigma_s^i|^2\right] ds + \int_0^t \sigma_s^i dW_s\right), \qquad i = 1, \ldots, m. \tag{2.5}$$

The next step is to introduce the *market price of risk process* $\{\lambda_t^\alpha\}$ which is defined by the equation

$$\mu_t^i = r_t - \Delta_t^i + \sum_{\alpha=1}^{n} \sigma_t^{i\alpha} \lambda_t^\alpha, \qquad i = 1 \ldots, m. \tag{2.6}$$

Here we note that $\{\lambda_t^\alpha\}$ has the same dimensionality as the Brownian motion driving the market. Under the condition of no arbitrage all risky assets have the same market price of risk process. Another name we will be using for this process is the relative risk process. We now define the *relative risk density process* $\{\rho_t\}$ as the solution of the equation

$$d\rho_t = -\rho_t \lambda_t \cdot dW_t, \tag{2.7}$$

which is

$$\rho_t = \exp\left(\frac{1}{2}\int_0^t |\lambda_s|^2 ds - \int_0^t \lambda_s \cdot dW_s\right). \tag{2.8}$$

A basic assumption in this investigation is that $\{\rho_t\}$ is a martingale.

The fact that $\{\rho_t\}$ is a martingale means that we can switch to another probability measure Q_T on the given probability space such that $\{\rho_T\}$ is the related Radon-Nikodym derivative

$$\frac{dQ_T}{dP} = \rho_T. \tag{2.9}$$

The next step is to apply Girsanov's theorem and then we see that the process

$$W_t^{\lambda} = W_t + \int_0^t \lambda_s ds, \tag{2.10}$$

is an n–dimensional Brownian motion with respect to the probability measure Q_T. We can summarise by arguing that arbitrage is not allowed in the market if there exists a relative risk process $\{\lambda_t^{\alpha}\}$ such that the relative risk density $\{\rho_t\}$ is a martingale. For non–dividend–paying assets it is a simple exercise to prove that the quantity $\rho_t S_t^i / B_t$ is a P–martingale. Again here, see e.g. Hughston (1996), Hunt and Kennedy (2000), and Karatzas and Shreve (2001) for a detailed analysis. This result is of importance in finance and in option pricing. A consequence is that the quantity S_t^i / B_t is a Q–martingale over the time horizon $[0, T]$.

Now we concentrate on the issue of option pricing, following the spirit of the discussion so far. In addition to the underlying assets S_t^i and the money market account, we consider the existence of a contingent claim in the market that has a payoff H_T for some maturity date T. Here H_T is an $\mathcal{F}_T$–measurable random variable in L^1 representing the random payoff of the claim. For example this could be a simple call option, an Asian option etc. In a complete market setting, one obtains a rational price for the contingent claim by constructing a portfolio that replicates the derivative, in other words it has exactly the same payoff. Whether or not the market is complete, however, the final expression for derivatives pricing takes the following form:

$$\frac{\rho_t H_t}{B_t} = E\left[\frac{\rho_T H_T}{B_T} | \mathcal{F}_t\right], \quad \text{for all} \quad t \in [0, T]. \tag{2.11}$$

We can call this formula a *price forecasting relation for derivatives.* As a result, the price of the derivative at any time $0 \leq t \leq T$ is

$$H_t = \frac{B_t}{\rho_t} E\left[\frac{\rho_T H_T}{B_T}\Big|\mathcal{F}_t\right], \tag{2.12}$$

which can be written also as

$$H_t = B_t E^{Q_T}\left[\frac{H_T}{B_T}\Big|\mathcal{F}_t\right]. \tag{2.13}$$

The final remark before we conclude this brief introduction concerns the special asset B_t/ρ_t. The process $\{\rho_t\}$ is dimensionless and $\{B_t\}$ is an asset price process. Therefore $\xi_t = B_t/\rho_t$ is also an asset, with some very interesting properties. In particular, by virtue of Itô's lemma we see that

$$d\xi_t = \xi_t(r_t + |\lambda_s|^2)dt + \xi_t\lambda_t \cdot dW_t. \tag{2.14}$$

It can be shown that one can construct a self–financing portfolio which consists of the money market account and the basic risky assets, and whose value process is $\{\xi_t\}$. This portfolio has the property that any asset in the market, when quoted in units of ξ_t is a P–martingale, i.e.

$$\frac{S^i_t}{\xi_t} = E\left[\frac{S^i_T}{\xi_T}\Big|\mathcal{F}_t\right], \quad \text{for} \quad t \leq T. \tag{2.15}$$

We call this asset the *natural numeraire process.* Here we observe that this equation is the same relation as (2.11), and therefore as far as price trajectories are concerned, assets and derivatives are essentially indistinguishable for derivatives with a positive payoff. Another process of interest is the inverse of the numeraire portfolio whose process we denote by $\{V_t\}$ and this is the process we call the *pricing kernel*:

$$V_T = \frac{1}{\xi_T}. \tag{2.16}$$

This process plays a fundamental role in the development of the present work, as will become apparent in later chapters. For now, we note that (2.12) becomes

$$H_t = E\left[\frac{V_T}{V_t} H_T\Big|\mathcal{F}_t\right]. \tag{2.17}$$

2.2 Interest rate modelling

In this section we briefly review the existing literature of term structure modelling. There are many great treatments of this subject; see e.g. Brigo and Mercurio (2001), Hunt and Kennedy (2000), Musiela and Rutkowski (1997), Pelsser (1998). Interest rate theory lies at the core of mathematical finance. It is of great practical importance and at the same time offers a framework of advanced intellectual stimulation. Most financial institutions have one or more 'fixed income desks' where complex derivative products are traded, and the volume of these transactions is impressive. There is a major difference between the modelling of interest rates and the general asset theory presented in the previous section. In term structure modelling the underlying asset is the *pure discount bond.* This is a product that guarantees the holder a fixed amount of cash at a specified future time T. The fixed amount is usually normalised to one unit of currency, and by guaranteed we mean that the product is default free. To acquire this contract one has to pay a specific premium and thereafter there is no obligation from the holder. We denote this price as $\{P_{tT}\}$, it is the price process of a pure discount bond that matures at time T quoted at time $t \in [0, T]$. Now the big difference with the standard model presented in the previous section is that there is a *continuum* of assets in this case, as opposed to a set of finitely many assets. A discount bond can be defined for each maturity date, therefore constructing the *term structure of interest rates.* Is this a realistic situation? In other words are there really discount bonds for *any* maturity date in the market? The answer is no. One might find discount bonds of a variety of maturities but not for all maturities. Still, this is not a major problem, at least from our modelling point of view. Even if we do not observe all maturity dates in the market, we *assume* that they exist. Simply because no one is offering to buy or sell them, this does not mean that they are not there: the key point is that they can be *defined.*

There is a number of interest rates that can be defined. The observed rates are the LIBOR and the swap rates. LIBOR stands for *London Interbank Offer Rate.* It is the rate under which one bank in London is willing to loan money to another bank. The LIBOR is an observed rate, it is quoted in the market. We denote this by L_{tT}.

Now let us consider the short rate with price process $\{r_t\}$, it is the interest rate that applies for a very short period of time. The instantaneous forward rate f_{tT} is the rate of interest contracted at time t for a very short period loan at some later time T. It is also called the *forward short rate.* The connection between the forward short rates and the discount bonds is given by the relation

$$f_{tT} = -\frac{\partial}{\partial T}\left(\ln P_{tT}\right). \tag{2.18}$$

In essence this is the *definition* of the instantaneous forward rates. One can invert this relation to obtain the explicit expression for the discount bond system as a function of the forward short rates:

$$P_{tT} = \exp\left(-\int_t^T f_{tu}du\right). \tag{2.19}$$

What this means is that if the discount bond system is differentiable with respect to the maturity date, then the modelling of discount bonds and forward rates is essentially indistinguishable. This observation plays a role for the remaining of this discussion.

To this end we should note that despite the differences with the finite assets case, the probabilistic background here is the same. We still consider the underlying filtration to be the natural augmented filtration of an n–dimensional Brownian motion. The underlying probability space $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, P)$ is where the Brownian motion lives. In addition, before we commence the brief presentation of existing term structure models, it should be mentioned that despite the great effort by many leading researchers during the last twenty years, no model can yet claim that is superior to all others, no model can be considered as the 'standard' model for interest rate modelling. This is being said not to vitiate the models that exist to date, rather to emphasise the common agreement in the community of the lack of a simple model or framework that is best in terms of intuition, generality, and practicality. To proceed, we consider the techniques that are available to model the term structure. A detailed discussion can be found in many textbooks, for example Hunt and Kennedy (2000). Here we consider the discount bonds to be the essential underlying elements of interest rate theory. One can, alternatively, model the short rates, or forward rates, or one can consider the so called 'market models' such as the BGM model (Brace, Gatarek and Musiela 1997), and swap rate models. Historically it was the work of Vasicek (1977) that was the starting point of modern interest rate theory. In this work one models the short rate by use of the stochastic equation

$$dr_t = (\theta - \alpha r_t)dt + \sigma dW_t, \tag{2.20}$$

for some constants θ, α, and σ. This is an example of a short rate model. In general we define such a model not only by the specification of a stochastic equation for the short rates, but also by the specification of some functional form for the risk premium process $\{\lambda_t\}$. Usually the stochastic differential equation for the short rates in such models will be a diffusion:

$$dr_t = b(t, r_t)dt + \sigma(t, r_t) \cdot dW_t, \tag{2.21}$$

and the risk premium will be specified by a relation of the form

$$\lambda_t = \lambda(t, r_t), \qquad t \geq 0. \tag{2.22}$$

The Vasicek model, though highly tractable, does not offer enough free parameters for calibration. That is to say, one cannot choose values for the three parameters such that a connection with all discount bonds for all maturities can be established. This is what essentially led Hull and White (1993) to introduce the so–called extended Vasicek model, replacing the constant parameters θ, α and σ with deterministic functions:

$$dr_t = (\theta_t - \alpha_t r_t)dt + \sigma_t dW_t. \tag{2.23}$$

As a conclusion we mention that this model allows for negative interest rates, which we will see here as something that needs to be avoided. Other short rate models include the Black–Karasinsky model (1991), and the Cox–Ingersoll–Ross model (1985). The latter is one of the few, apart from the work of Hughston and his collaborators, that guarantees positive interest rates. The scheme we propose in later chapters of this thesis is also a positive interest rate framework.

Another way to model the term structure is to consider the forward rates instead. This is the monumental work of Heath, Jarrow and Morton (1992), hereafter HJM. The idea of the HJM framework is to model the process $\{f_{tT}\}_{0\leq t\leq T\leq T^*}$ by providing a stochastic equation that is satisfied by the forward rates:

$$df_{tT} = \mu_{tT}dt + \sigma_{tT} \cdot dW_t. \tag{2.24}$$

Here T^* is some fixed time horizon over which the model is specified. To establish the connection with the discount bonds, not only is it necessary to consider the differentiability with respect to the maturity date but also to allow sufficient enough technical conditions to hold so that one can obtain the relevant stochastic equation for discount bonds.

There are many advantages to be gained by directly modelling of the discount bonds. These are the assets that are observable in the market and any other model is obliged to be a subset of a discount bond model. One can argue that any condition that arises in a pure discount bond model will in that context be expressed in a somewhat more natural and intuitive nature. For example, the positivity of interest rates is grasped in the following two conditions:

$$0 < P_{tT} \leq 1, \qquad \text{for all} \qquad 0 \leq t \leq T < \infty, \tag{2.25}$$

and

$$\frac{\partial P_{tT}}{\partial T} < 0. \tag{2.26}$$

For many models these conditions are not satisfied, leading to the possibility of negative interest rates, and hence arbitrage opportunities.

In this spirit we now consider the so–called Flesaker–Hughston class of models. This is a very general family of models in which positivity is integrated from the beginning. In this framework the discount bond system takes the following form:

$$P_{tT} = \frac{\int_T^\infty M_{ts} ds}{\int_t^\infty M_{ts} ds}, \tag{2.27}$$

where $\{M_{ts}\}_{0 \le t \le s < \infty}$ is a family of positive $\{\mathcal{F}_t\}$–martingales, under any measure that is locally equivalent to the natural measure P. The family of martingales has to be jointly measurable and the integral needs to be finite almost surely. The best known model within this framework is the so–called *rational log–normal model*, for which P_{tT} is written as the following quotient

$$P_{tT} = \frac{A_T + B_T M_t}{A_t + B_t M_t}. \tag{2.28}$$

In this case $\{M_t\}$ is a unit initialised log–normal martingale, and A_t and B_t are absolutely continuous deterministic functions, positive and decreasing. The initial term–structure is then given by

$$P_{0T} = \frac{A_T + B_T}{A_0 + B_0}. \tag{2.29}$$

This model has the striking feature that allows for explicit formulae for caps and swaptions; see Flesaker and Hughston (1996) and Musiela and Rutkowski (1997). In addition there are upper and lower bounds for the discount bonds, and for interest rates. We are therefore left with a framework that directly models the discount bonds, guarantees positive and non–explosive interest rates, and prices explicitly caps and swaptions. For further details see Flesaker and Hughston (1996a, 1998), and Rutkowski (1997). In later chapters this model will be mentioned again and connections with new results will be established.

Chapter 3

Chaotic approach to interest rate modelling

3.1 Axiomatic framework for asset price dynamics

In this chapter we describe the foundations of the chaotic approach to term structure modelling. We begin with a brief review of the theoretical framework within which we examine the dynamics of interest rates. The idea is to develop an axiomatic scheme that will ensure the existence of an arbitrage–free system of discount bonds over all time horizons, but that is general enough also to allow a place for other systems of assets. The methodology proposed here, which in effect unifies a number of important features of the theory of interest rate modelling and the theory of volatility modelling, is based on a conditional variance representation for the pricing kernel and makes use of the Wiener chaos expansion technique in a novel way. The axiomatic framework presented here is such that many well known stochastic equations for asset pricing follow as a consequence. That is to say, we do not *assume* the usual no–arbitrage equations for assets but these follow from the axioms. In addition we try to include as much economic intuition as possible in the formulation of the axioms.

Let us first describe briefly the relevant probabilistic background. We model the unfolding of random market events in the usual way with the specification of a fixed probability space $(\Omega, \mathcal{F}, P)$ which we denote as Π. We assume that Π is equipped with the standard augmented filtration $\Phi = \{\mathcal{F}_t\}_{0 \leq t < \infty}$ generated by a system of one or more independent Wiener processes $\{W_t^\alpha\}_{0 \leq t < \infty}$ $(\alpha = 1, \cdots, k)$. The probability measure P is to be interpreted as the 'natural' measure, and filtration–dependent concepts (such as adaptedness or the martingale property) are defined relative to Φ. We assume in this investigation that the random processes on Π followed by asset prices are continuous, and are given by Itô processes.

The absence of arbitrage in the economy will be characterised according to the following scheme. Firstly, we assume the existence of a non–dividend–paying asset with price process $\{\xi_t\}$, adapted to Φ, which we call the 'natural numeraire', satisfying $\xi_t > 0$ for all $t \in [0, \infty)$. We first mentioned this asset in chapter 2. The natural numeraire asset is also referred to as the Long portfolio since it was Long (1990) who introduced it and considered its properties. Indeed, we consider the main properties of this asset as integral part of our axiomatic framework.

We now give the axioms:

(A1) There exists a strictly increasing (and hence 'risk–free') asset with absolutely continuous price process $\{B_t\}$, which we call the money–market account.

(A2) If $\{S_t\}$ is the price–process of any asset, and $\{D_t\}$ is the adapted dividend rate process for that asset, so that $D_t dt$ represents the small random dividend paid at time t, then the process $\{M_t\}$ defined by

$$M_t = \frac{S_t}{\xi_t} + \int_0^t \frac{D_s}{\xi_s} ds, \tag{3.1}$$

is a martingale.

(A3) There exists an asset (a floating rate note) that offers a dividend rate sufficient to ensure that the value of the asset remains constant.

These are the axioms that one needs to consider in order to derive a consistent theory of asset pricing in the case of price processes adapted to a Brownian filtration. This will be demonstrated in the present and the following chapters.

Let us start by examining some of the consequences of these axioms. Since the process $\{B_t\}$ introduced in (A1) is by assumption absolutely continuous and strictly increasing, there exists an adapted, positive process $\{r_t\}$ such that

$$B_t = B_0 \exp\Big(\int_0^t r_s ds\Big). \tag{3.2}$$

Since the money market account is a non–dividend–paying asset, it follows as a consequence of (A1) and (A2) that there exists a positive martingale $\{\rho_t\}$ such that

$$\frac{B_t}{\xi_t} = \rho_t. \tag{3.3}$$

Because $\{\rho_t\}$ is positive, there exists an adapted vector–valued process $\{\lambda_t\}$ such that

$$d\rho_t = -\rho_t \lambda_t dW_t. \tag{3.4}$$

Here, and similarly elsewhere, we use the shorthand

$$\lambda_t dW_t = \sum_{\alpha=1}^{k} \lambda_t^\alpha dW_t^\alpha. \tag{3.5}$$

As a consequence of (3.4), we then have

$$\rho_t = \rho_0 \exp\Big(-\int_0^t \lambda_s dW_s - \frac{1}{2}\int_0^t \lambda_s^2 ds\Big). \tag{3.6}$$

It is important to note here that at most one process $\{B_t\}$ can exist satisfying axioms (A1) and (A2). This is demonstrated as follows. If $\{B_t^*\}$ were another such increasing price process, then we would have

$$\frac{\rho_t}{B_t} = \frac{\rho_t^*}{B_t^*}, \tag{3.7}$$

for some positive martingale $\{\rho_t^*\}$. But this relation implies that

$$\frac{d\rho_t}{\rho_t} = (r_t - r_t^*)dt + \frac{d\rho_t^*}{\rho_t^*}, \tag{3.8}$$

which shows that for $\{\rho_t\}$ and $\{\rho_t^*\}$ both to be martingales we have $\{r_t\} = \{r_t^*\}$. This result is important since, from an economic point of view, only one money market account is allowed in a 'domestic' economy. The multicurrency situation is of course somewhat different and we consider this case later.

3.2 Non–dividend paying assets

By use of axiom (A2) we now demonstrate how to derive the usual no–arbitrage dynamics, both in the case of an asset that is of limited liability and also in the more general case of any asset in the market. Let us first consider an asset with price process $\{S_t\}$ that pays no dividends. Then, making use of (A2) we have:

$$S_t = \frac{B_t M_t}{\rho_t}, \tag{3.9}$$

where $\{M_t\}$ is a martingale. Thus, if we write $dM_t = \theta_t dW_t$ it is a straightforward exercise to verify that

$$dS_t = (r_t S_t + \lambda_t \psi_t)dt + \psi_t dW_t, \tag{3.10}$$

where the vector–valued process $\{\psi_t\}$ is defined by

$$\psi_t = \frac{B_t \theta_t}{\rho_t} + \lambda_t S_t. \tag{3.11}$$

In particular, if the asset price process $\{S_t\}$ is positive, then $\{M_t\}$ is positive, and we can write $\theta_t = (\nu_t - \lambda_t)M_t$ for some vector–valued process $\{\nu_t\}$, from which it follows that $\psi_t = \nu_t S_t$. In that case the dynamical equation satisfied by the process $\{S_t\}$ can be written in the form

$$\frac{dS_t}{S_t} = (r_t + \lambda_t \nu_t)dt + \nu_t dW_t, \tag{3.12}$$

and thus it is apparent that $\{\nu_t\}$ is the adapted vector–valued volatility process for the given asset, and $\{\lambda_t\}$ has the interpretation of the market risk premium process. We recognise (3.12) as the dynamics of a risky asset with limited liability in a market with no arbitrage. On the other hand, the dynamical equation (3.10) has the advantage of holding in the more general situation for assets such as portfolio positions including borrowing, short sales, or derivatives, where the value of the position may swing into the red as well as the black.

3.3 Dividend–paying assets

In the case of a dividend–paying asset these formulae need to be modified slightly, and in place of (3.10) we obtain

$$dS_t = (r_t S_t - D_t + \lambda_t \psi_t)dt + \psi_t dW_t, \tag{3.13}$$

as a consequence of (A2), with ψ_t defined as before according to (3.11). To see this consider axioms (A1) and (A2). Again here the natural numeraire can be written as

$$\xi_t = \frac{B_t}{\rho_t}, \tag{3.14}$$

where the martingale $\{\rho_t\}$ follows the dynamics (3.4). Now we have that

$$M_t = \frac{S_t}{B_t}\rho_t + \Gamma_t, \tag{3.15}$$

is a martingale, where we define

$$\Gamma_t = \int_0^t \frac{D_s}{B_s}\rho_s ds. \tag{3.16}$$

In other words the asset price process $\{S_t\}$ is now represented as

$$S_t = \frac{B_t}{\rho_t}\left(M_t - \Gamma_t\right). \tag{3.17}$$

By taking the differential of the above expression we have that

$$dS_t = d\left(\frac{B_t}{\rho_t}\right)\left(M_t - \Gamma_t\right) + \frac{B_t}{\rho_t} d\left(M_t - \Gamma_t\right) + d\left(\frac{B_t}{\rho_t}\right) d\left(M_t - \Gamma_t\right). \tag{3.18}$$

The dynamics for $\{B_t/\rho_t\}$ are easily calculated and in addition we have that

$$d\Big(M_t - \Gamma_t\Big) = -\frac{D_t}{B_t}\rho_t dt + \theta_t dW_t, \tag{3.19}$$

where recall that $dM_t = \theta_t dW_t$. After performing the calculations we end up with the result (3.13) where the vector–valued process $\{\psi_t\}$ is defined in (3.11). Then if $\{S_t\}$ is positive we can introduce a proportional dividend rate process $\{\delta_t\}$ by the relation $D_t = \delta_t S_t$, and we obtain the simplified expression

$$\frac{dS_t}{S_t} = (r_t - \delta_t + \lambda_t \nu_t)dt + \nu_t dW_t, \tag{3.20}$$

where the process $\{\nu_t\}$ is defined as before by $\psi_t = \nu_t S_t$. Clearly, (3.20) conforms to the familiar dynamics of a dividend or interest paying asset with limited liability. For example, if $\{S_t\}$ is the price process of a foreign currency, then $\{\delta_t\}$ corresponds to the overnight rate process for that currency. We consider this case in more detail in chapter 6.

3.4 Floating rate notes

Now let us consider the consequences of axiom (A3) in more detail. Such an asset that maintains a constant value has the interpretation of being a floating rate note. Equation (3.20) shows that if we set $S_t = 1$ for all $t \in [0, \infty)$ then the 'dividend' rate offered by this instrument must be the short rate process $\{r_t\}$. It follows that

$$\frac{1}{\xi_t} + \int_0^t \frac{r_s}{\xi_s} ds \qquad \text{is a martingale.} \tag{3.21}$$

In particular since $\{r_t\}$ and $\{\xi_t\}$ are positive processes we deduce that

$$E\left[\frac{1}{\xi_t}\right] < \infty, \tag{3.22}$$

and

$$E\left[\int_0^t \frac{r_s}{\xi_s} ds\right] < \infty, \tag{3.23}$$

for all $t \in [0, \infty)$. The significance of these relations will be discussed shortly. It is here worth noting that the axiomatic framework considered in this section in connection with the techniques we are going to use later in this chapter, has further implications in the general asset pricing theory. We consider these in chapter 5.

3.5 Price processes for discount bonds

To proceed further we introduce a system of discount bonds on Π. This will be the discount bond system associated with the base currency in terms of which the other assets on Π are priced and with respect to which the money market process $\{B_t\}$ is defined. The discount bond price processes will be denoted $\{P_{tT}\}$, where $0 \leq t \leq T < \infty$.

We shall as usual regard the zero–coupon bond for a given value of T as a default–free contract that pays one unit of the base currency at time T. Then P_{tT} denotes the price of the bond at time t, and by the definition of the contract we require that $P_{TT} = 1$ for all $T \in [0, \infty)$. For the moment we make no other assumptions concerning the discount bond processes other than those properties applicable to all assets implicit in axioms (A1), (A2), and (A3), though later we add a further important assumption concerning the asymptotic behaviour of the bond prices in the case of an infinite time horizon.

Since $\{P_{tT}\}$ represents the price process of a non–dividend–paying asset for each value of $T \in [0, \infty)$, it follows from (A2) that $\{P_{tT}/\xi_t\}$ is a martingale, and hence that there exists a family of positive martingales $\{M_{tT}\}$ such that

$$P_{tT} = \frac{B_t M_{tT}}{\rho_t}. \tag{3.24}$$

Because $\{M_{tT}\}$ is a positive martingale for each bond maturity date $T \in [0, \infty)$, there exists a vector–valued process $\{\Omega_{tT}\}$ such that

$$\frac{dM_{tT}}{M_{tT}} = (\Omega_{tT} - \lambda_t) dW_t. \tag{3.25}$$

We then deduce from equations (3.2), (3.4), (3.24) and (3.25) that the dynamics of the discount bond system are given by

$$\frac{dP_{tT}}{P_{tT}} = (r_t + \lambda_t \Omega_{tT}) dt + \Omega_{tT} dW_t. \tag{3.26}$$

We thus recognise $\{\Omega_{tT}\}$ as being the T–maturity discount bond vector relative volatility process. It then follows, by integrating (3.26), if we make use of the relation $P_{tt} = 1$, that the discount bond price processes can be represented in the form

$$P_{tT} = P_{0tT} \frac{\exp\left(\int_0^t \lambda_s \Omega_{sT} ds + \int_0^t \Omega_{sT} dW_s - \frac{1}{2}\int_0^t \Omega_{sT}^2 ds\right)}{\exp\left(\int_0^t \lambda_s \Omega_{st} ds + \int_0^t \Omega_{st} dW_s - \frac{1}{2}\int_0^t \Omega_{st}^2 ds\right)}. \tag{3.27}$$

The money market account process is then given by a corresponding expression of the form

$$B_t = \frac{B_0}{P_{0t} \exp\left(\int_0^t \lambda_s \Omega_{st} ds + \int_0^t \Omega_{st} dW_s - \frac{1}{2}\int_0^t \Omega_{st}^2 ds\right)}. \tag{3.28}$$

Here we have used the notation

$$P_{0tT} = \frac{P_{0T}}{P_{0t}}, \tag{3.29}$$

for the t–forward price made at time 0 for a T–maturity discount bond. Formulae (3.27) and (3.28) are well known and can be found, for example, in Hughston (1996) and Flesaker and Hughston (1997).

3.6 The volatility structure approach

An interesting feature of expressions (3.27) and (3.28) is that the discount bond system and the money market account can be represented directly in terms of the market risk premium process $\{\lambda_t\}$ and the bond volatility process $\{\Omega_{tT}\}$, together with the initial discount function P_{0t}, without direct reference to the short rate $\{r_t\}$. It is therefore legitimate to regard the processes $\{\lambda_t\}$ and $\{\Omega_{tT}\}$ as being subject to an exogenous specification. Indeed, historically this observation is of considerable significance since it forms the basis of the approach to interest rate derivatives pricing frequently used in practice according to which one 'models the volatility structure'. In such an approach one typically assumes market completeness, then transforms to the risk neutral measure to eliminate the market risk premium, and then models the bond volatility process exogenously, calibrating it to a suitable given set of market interest rate option data. It has been a problematic feature of the volatility approach, however, that if $\{\lambda_t\}$ and $\{\Omega_{tT}\}$ are specified exogenously, then there is no guarantee that axiom (A1) is satisfied—that is to say, the resulting interest rates need not be positive. Additionally, there is no reason to suppose, *a priori*, that the bond volatilities will take on a given form in the risk neutral measure.

3.7 Martingale approach to discount bond dynamics

Let us therefore put to one side the 'volatility structure' approach, and return to the consideration of the assumptions (A1), (A2), and (A3) in the context of a term structure model. Since the discount bonds are non–dividend–paying assets, it follows as a consequence of (A2) that the martingale relations

$$E\left[\frac{P_{tT}}{\xi_t}\right] < \infty, \tag{3.30}$$

and

$$\frac{P_{tT}}{\xi_t} = E_t\left[\frac{P_{uT}}{\xi_u}\right], \tag{3.31}$$

hold for all $0 \leq t \leq u \leq T < \infty$. Here $E_t[-]$ denotes as usual the conditional expectation with respect to the σ–algebra $\mathcal{F}_t$. It follows from (3.30) by setting $t = T$ that the existence of the discount bond system implies that the inequality (3.22) holds for all $t \in [0, \infty)$. On the other hand, as was shown by Baxter (1997), the inequality (3.23) is the assumption required to ensure differentiability of the bond price system with respect to the maturity date, i.e., to ensure that there exists a family of Itô processes $\{f_{tu}\}_{0\leq t\leq u<\infty}$, such that

$$P_{tT} = \exp\left(-\int_t^T f_{tu}du\right). \tag{3.32}$$

It then follows that

$$-\frac{\partial}{\partial T}\Big(\ln P_{tT}\Big) = f_{tT}, \tag{3.33}$$

and also that

$$\lim_{t\to T} f_{tT} = r_T, \tag{3.34}$$

and

$$\lim_{t\to T} \Omega_{tT} = 0. \tag{3.35}$$

The significance of the existence of the instantaneous forward rates is that the class of interest rate models under consideration here includes all positive interest HJM models (Heath, Jarrow and Morton 1992) defined over the relevant time horizon. Although all positive HJM models are elements of the family of models considered here, the present framework is more general since we make no assumption of market completeness, a basic element of the HJM approach.

In what follows here the instantaneous forward rates play essentially a secondary role, and primary significance is attached to modelling the process $\{\xi_t\}$.

In particular, setting $u = T$ in (3.31) we obtain the pricing formula

$$P_{tT} = \xi_t E_t\left[\frac{1}{\xi_T}\right]. \tag{3.36}$$

Thus, once axioms (A1), (A2), and (A3) have been specified, the associated discount bond system is also determined. We note that P_{tT} is unchanged if we multiply ξ_t by a positive constant.

3.8 Dynamics of the pricing kernel

We are now in a position to introduce the pricing kernel process by means of the expression

$$V_t = \frac{1}{\xi_t}. \tag{3.37}$$

The intuition behind this process is that it represents the value of one unit of cash in units of the natural numeraire. As we will demonstrate shortly, it plays a fundamental role in the discussion to follow. It follows from equation (3.2) that $V_t = \rho_t/B_t$, and from (3.22) we have $E[V_t] < \infty$ for all $t \in [0, \infty)$. In particular, since B_t is $\mathcal{F}_t$–measurable and increasing, and using the fact that $\{\rho_t\}$ is a martingale, we deduce that

$$E_t\left[V_T\right] = E_t\left[\frac{\rho_T}{B_T}\right] < E_t\left[\frac{\rho_T}{B_t}\right] = \frac{E_t\left[\rho_T\right]}{B_t} = \frac{\rho_t}{B_t} = V_t, \tag{3.38}$$

for $0 \leq t < T < \infty$, and hence that $\{V_t\}$ is a supermartingale.

Now writing (3.36) in the form

$$P_{tT} = E_t\left[\frac{V_T}{V_t}\right], \tag{3.39}$$

we see that $\{P_{tT}\} < 1$ for all $0 \leq t < T < \infty$. The first to introduce the pricing kernel process as a fundamental modelling element was Constantinides in 1992. Suppose that $\{H_t\}$ is for $0 \leq t \leq T < \infty$ the price process of a derivative asset on Π with a European–style payoff H_T at time T. Then by (A2) we have

$$H_t = E_t\left[\frac{V_T}{V_t} H_T\right], \tag{3.40}$$

a relation that remains valid independently of any hedgeability considerations. Note that no assumption of market completeness is made in our axiomatic scheme.

Now let us examine more closely some of the properties of the pricing kernel. It follows from equations (3.2), (3.3) and (3.4) that the dynamics of $\{V_t\}$ are given by

$$dV_t = -r_t V_t dt - \lambda_t V_t dW_t. \tag{3.41}$$

Therefore, given $\{V_t\}$ we can recover the short rate $\{r_t\}$ and the market risk premium process $\{\lambda_t\}$. Integrating (3.41) from 0 to t we get

$$V_t = V_0 - \int_0^t r_s V_s ds - \int_0^t \lambda_s V_s dW_s. \tag{3.42}$$

Now, axiom (A3) implies that the process

$$V_t + \int_0^t r_s V_s ds, \tag{3.43}$$

is a martingale. It follows then from (3.42) that the local martingale

$$N_t = \int_0^t \lambda_s V_s dW_s, \tag{3.44}$$

is a true martingale. The martingale relation

$$E_t \left[V_T + \int_0^T r_s V_s ds \right] = V_t + \int_0^t r_s V_s ds, \tag{3.45}$$

implies that

$$E_t [V_T] = V_t - E_t \left[\int_t^T r_s V_s ds \right]. \tag{3.46}$$

Dividing by V_t we arrive at the formula

$$P_{tT} = 1 - E_t \left[\int_t^T \frac{V_s}{V_t} r_s ds \right], \tag{3.47}$$

which has a natural economic interpretation from which a number of interesting consequences can be deduced. It follows for example as a corollary of (3.47) that for any two maturity dates T_1 and T_2 we have

$$P_{tT_1} - P_{tT_2} = E_t \left[\int_{T_1}^{T_2} \frac{V_s}{V_t} r_s ds \right]. \tag{3.48}$$

Therefore if $T_2 > T_1$, we deduce that $\{P_{tT_2}\} < \{P_{tT_1}\}$, and hence that the random forward price $\{P_{tT_1T_2}\} = \{P_{tT_2}/P_{tT_1}\}$, made at time t for purchase at time T_1 of a T_2–maturity discount bond satisfies $0 < \{P_{tT_1T_2}\} \leq 1$ for all $0 \leq t \leq T_1 \leq T_2 < \infty$. This in turn implies the positivity of all forward rates.

Another interesting corollary of (3.47) follows if we differentiate each side of this equation with respect to T, from which we deduce that

$$f_{tT}P_{tT} = E_t\left[\frac{V_s}{V_t}r_T\right]. \tag{3.49}$$

This relation shows that the instantaneous forward rate can be interpreted as the value, at time t, future–valued to time T, of the contingent claim that pays the short rate r_T at time T on a unit principal. It follows that the term structure density $\rho_t(x)$ for tenor $x = T - t$ (Brody and Hughston 2001a,b, 2002) is the value at time t of an instrument that pays the rate r_T at time T on a unit principal.

Equation (3.47) says that ownership of a T–maturity discount bond is equivalent to ownership of one unit of the cash asset, but without the right to the dividend flow of the cash asset from time t to time T. Or to put the matter in another way, a money–lender will be willing at time t to part with one unit of cash in exchange for a discount bond maturing at time T together with a continuous flow of interest from time t to time T. Equivalently, to hold a T–maturity floating–rate note is the same as holding a T–maturity discount bond together with the right to a continuous stream of interest from time t to time T.

3.9 The conditional variance representation

Now suppose we consider the case of an interest rate system in an economy with an infinite time horizon. It follows from (3.47) that

$$P_{0T} = 1 - E\left[\int_0^T \frac{V_s}{V_0}r_s ds\right]. \tag{3.50}$$

This relation can be interpreted as saying that the value of a T–maturity discount bond at time 0 is one unit of cash less the present value of the interest stream from time 0 to time T.

The idea is that by holding the discount bond one forgoes the dividends associated with the cash until the maturity date of the bond—at which point one acquires the cash. The ownership of a discount bond that never matures (i.e. matures at $T = \infty$) is equivalent to ownership of a unit of floating rate note stripped of its interest stream for all time—in other words, the ownership of nothing. As a consequence we conclude that for a consistent economic framework we must assume that

$$\lim_{T\to\infty} P_{tT} = 0, \tag{3.51}$$

almost surely for all $t \in [0, \infty)$. Indeed, we can now take it as part of the definition of a discount bond system that it should satisfy condition (3.51). It follows from (3.39) that (3.51) holds only if

$$\lim_{T\to\infty} E\left[V_T\right] = 0. \tag{3.52}$$

This is the condition that the process $\{V_t\}$ is a 'potential', i.e. a positive supermartingale with the property that its expectation vanishes in the limit. Thus, as was pointed out by Rogers (1997), it should be regarded as an essential element of interest rate theory that the pricing kernel process should have this property. Accordingly, we formalise these considerations by including the following additional axiom in our economic framework:

(A4) There exists a system of discount bond price processes $\{P_{tT}\}_{0\leq t\leq T<\infty}$ with the property $\lim_{T\to\infty} P_{tT} = 0$.

We now proceed to deduce a relation that plays an important role in what there is to follow. First, note that equation (3.47) after the use of axiom (A4) becomes

$$V_t = \lim_{T\to\infty} E_t\left[\int_t^T r_s V_s ds\right]. \tag{3.53}$$

We will now make use of both the martingale convergence theorem, to prove that the integral has a limit, and of the monotone convergence theorem to interchange the order of the limit and the expectation. First, note that since the process $\{N_t\} = \{V_t + \int_0^t r_s V_s ds\}$ is a positive martingale, by use of the L^1 version of the martingale convergence theorem it follows that this process has a limit, which we will call

$$A_\infty = \int_0^\infty r_s V_s ds. \tag{3.54}$$

In other words we have

$$\lim_{t\to\infty} A_t = A_\infty. \tag{3.55}$$

Moreover, for this asymptotic random variable we have that

$$E\left[\int_0^\infty r_s V_s ds\right] < \infty, \qquad \text{almost surely.} \tag{3.56}$$

This is an important relation since, as will be shown shortly, it guarantees the existence of the elementary particle of the chaotic approach, i.e. the generator X_∞.

Consider now the fact that $\{A_t\}$ is a nondecreasing, positive process and therefore by use of the conditional form of the monotone convergence theorem it follows that we have

$$\lim_{T\to\infty} E_t\left[\int_t^T r_s V_s ds\right] = E_t\left[\lim_{T\to\infty}\int_t^T r_s V_s ds\right]. \tag{3.57}$$

As a result of the considerations above we finally derive the relation

$$V_t = E_t\left[\int_t^\infty r_s V_s ds\right]. \tag{3.58}$$

This formula, which is in essence a restatement of (A3), has the economic interpretation that a floating rate note that promises to pay the rate $\{r_t\}$ on a unit principal in perpetuity necessarily has the value unity.

An alternative expression for $\{V_t\}$ can be deduced from (3.58) if we consider again the increasing process

$$A_t = \int_0^t r_s V_s ds. \tag{3.59}$$

Then we obtain the relation

$$V_t = E_t [A_\infty] - A_t, \tag{3.60}$$

that forms the basis of the Flesaker–Hughston framework and its extensions (see, e.g., Flesaker and Hughston 1996, 1997, 1998, Rutkowski 1997, Musiela and Rutkowski 1997, Rogers 1997, Hunt and Kennedy 2000, Jin and Glasserman 2001).

In the present investigation we take an alternative point of view and emphasise a rather different feature of the pricing kernel that emerges in this context: namely, that $\{V_t\}$ can be interpreted as a *conditional variance.* More precisely, let us define the process $\{\sigma_t\}$ be an adapted vector–valued process satisfying

$$\sigma_t^2 = r_t V_t. \tag{3.61}$$

Then we can define the random variable X_∞ by the formula

$$X_\infty = \int_0^\infty \sigma_s dW_s. \tag{3.62}$$

In essence here, again, we mean the integral $X_\infty = \int_0^\infty \sigma_s^\alpha \cdot dW_s^\alpha$, for $\alpha = 1, \cdot, k$, and we use the Einstein notation for summation. The existence of X_∞ is guaranteed by virtue of axiom (A3) as was shown above. The result of the above considerations is the transformation of (3.58) into the following relation:

$$V_t = E_t \left[\int_t^\infty \sigma_s^2 ds \right]. \tag{3.63}$$

At this stage, based on the formula above we are going to use the Itô isometry in order to establish the conditional variance representation. We have:

$$V_t \;=\; E_t \left[\int_t^\infty \sigma_s^2 ds \right]$$

$$
\begin{aligned}
&= E_t\left[\left(\int_t^\infty \sigma_s dW_s\right)^2\right] \\
&= E_t\left[\left(\int_0^\infty \sigma_s dW_s - \int_0^t \sigma_s dW_s\right)^2\right]. \qquad (3.64)
\end{aligned}
$$

However since

$$E_t\left[X_\infty\right] = \int_0^t \sigma_s dW_s, \tag{3.65}$$

we deduce that:

$$V_t = E_t\left[\left(X_\infty - E_t\left[X_\infty\right]\right)^2\right], \tag{3.66}$$

which we recognise as the conditional variance of X_∞ with respect to the σ–algebra $\mathcal{F}_t$. In particular we note that $X_\infty \in L^2(\Omega, \mathcal{F}, P)$. From that point onwards, we consider this random variable as the 'primitive' ingredient of the theory, and we will call it the 'generator' of the associated interest rate model.

3.10 Interest rate models as elements of $L^2(\Omega, \mathcal{F}, P)$

Let us now recapitulate what we have learned so far. The market is characterised by a probability space $\Pi = (\Omega, \mathcal{F}, P)$ which without loss of generality we can assume to be the classical Wiener space associated with a system of n independent Brownian motions. If we assume the existence of an arbitrage–free system of discount bonds on Π then it follows from the considerations of the previous sections that there exists a random variable $X_\infty \in L^2(\Pi)$ with zero mean such that the pricing kernel process $\{V_t\}$ is given by the conditional variance

$$V_t = E_t\left[\left(X_\infty - E_t\left[X_\infty\right]\right)^2\right], \tag{3.67}$$

and the discount bond system is given by

$$P_{tT} = \frac{E_t\left[V_T\right]}{V_t}. \tag{3.68}$$

The pricing kernel is fully determined by the random variable X_∞. This means that the interest rate system is fully determined by X_∞ itself. In fact, each interest rate model corresponds to an equivalence class of elements of $L^2(\Pi)$, the relevant equivalence being under arbitrary random adapted orthogonal transformations of the vector process $\{\sigma_t\}$, in (3.62). This comes as a consequence of expression (3.63). We see that, in essence, what is needed to define the pricing kernel, and therefore the discount bond system is only a scalar, the length of the vector process $\{\sigma_t\}$. In other words, a different scalar will give rise to a different model, and therefore it suffices to consider only one element of the vector process $\{\sigma_t\}$, say the first component. As a result, in the case of a k–dimensional Wiener process where $k > 1$, the equivalence class can be represented by a single one–dimensional Wiener process, and all other components will only contribute to an additional amount of information. We can always therefore, for convenience, arrange that the integration in (3.62) is with respect to a specific choice of one of the Brownian motions, $\{W_t^\alpha\}(\alpha = 1, \cdots)$, and that the integrand is a scalar.

Conversely, given any random variable $X_\infty \in L^2(\Pi)$ with the property that X_∞ is not $\mathcal{F}_t$–measurable for any finite value of t, it follows immediately that the corresponding pricing kernel (3.66) satisfies $\{V_t\} > 0$ for all $t \in [0, \infty)$, and can be used to determine the short rate process $\{r_t\}$ and the money market account process $\{B_t\}$. One can then show that the associated local martingale $\{V_t B_t\}$ is a true martingale, and thus that axioms (A1)–(A4) are satisfied for the asset system consisting of the natural numeraire, the money market account, and the discount bonds.

To prove that the process $\{\rho_t\}$ defined by $\rho_t = V_t B_t$ is a true martingale, and hence that $\{B_t\}$ satisfies the conditions of (A2), it suffices to demonstrate that $E[\rho_t]$ is constant for all $t \in [0, \infty)$.

We regard X_∞, and hence $\{\sigma_t^2\}$, as given, and define the processes $\{r_t\}$ and $\{V_t\}$ by (3.61) and (3.64) respectively. Setting $B_0 = 1$ for convenience here, we then write

$$V_t = E_t\left[\int_t^\infty \sigma_s^2 ds\right], \tag{3.69}$$

and

$$B_t = \exp\left(\int_0^t \frac{\sigma_s^2}{V_s} ds\right). \tag{3.70}$$

Since B_t is $\mathcal{F}_t$–measurable, it follows from the tower property of conditional expectation that

$$\begin{aligned} E[\rho_t] &= E\left[E_t\left[\int_t^\infty \sigma_s^2 ds\right] \exp\left(\int_0^t \frac{\sigma_s^2}{V_s} ds\right)\right] \\ &= E\left[\int_t^\infty \sigma_s^2 ds \exp\left(\int_0^t \frac{\sigma_s^2}{V_s} ds\right)\right]. \end{aligned} \tag{3.71}$$

Since the bracketed quantity in (3.71) is differentiable, it follows, for $t_1, t_2 \in [0, \infty)$ that

$$\begin{aligned} E[\rho_{t_2} - \rho_{t_1}] &= E\left[\int_{t_1}^{t_2} d\left\{\int_t^\infty \sigma_s^2 ds \exp\left(\int_0^t \frac{\sigma_s^2}{V_s} ds\right)\right\}\right] \\ &= E\left[\int_{t_1}^{t_2} \left\{ -\sigma_t^2 \exp\left(\int_0^t \frac{\sigma_s^2}{V_s} ds\right)\right.\right. \\ &\quad \left.\left. + \int_t^\infty \sigma_s^2 ds \frac{\sigma_t^2}{V_t} \exp\left(\int_0^t \frac{\sigma_s^2}{V_s} ds\right)\right\} dt\right]. \end{aligned} \tag{3.72}$$

By use of Fubini's theorem we then interchange the integration and the expectation to obtain

$$E[\rho_{t_2} - \rho_{t_1}] = \int_{t_1}^{t_2} E\left[-\sigma_t^2 \exp\left(\int_0^t \frac{\sigma_s^2}{V_s} ds\right) + \int_t^\infty \sigma_s^2 ds \frac{\sigma_t^2}{V_t} \exp\left(\int_0^t \frac{\sigma_s^2}{V_s} ds\right)\right] dt. \tag{3.73}$$

By use again of the tower property, and (3.70), we deduce that

$$E[\rho_{t_2} - \rho_{t_1}] = \int_{t_1}^{t_2} E\left[-\sigma_t^2 \exp\left(\int_0^t \frac{\sigma_s^2}{V_s} ds\right) + E_t\left[\int_t^\infty \sigma_s^2 ds\right] \frac{\sigma_t^2}{V_t} \exp\left(\int_0^t \frac{\sigma_s^2}{V_s} ds\right)\right] dt \tag{3.74}$$

which vanishes identically.

We shall call such a system of assets $\{\xi_t, B_t, P_{tT}\}$ for $0 \leq t \leq T < \infty$ satisfying axioms (A1)–(A4) an interest rate model. It is implicit in this definition that interest rates are positive and that there are no–arbitrage opportunities.

For each choice of X_∞ satisfying the conditions noted above we thus obtain an interest rate model, and conversely given an interest rate model we can determine a corresponding element of $L^2(\Pi)$.

The Hilbert space $L^2(\Pi)$ has a rich structure that can be exploited in the analysis of the associated interest rate systems. The key point is that we can represent X_∞, and therefore characterise the corresponding interest rate system, by use of a *Wiener chaos expansion*. In particular, we obtain the following representation for the random variable X_∞:

$$X_\infty = \int_0^\infty \phi_s dW_s + \int_0^\infty \int_0^s \phi_{ss_1} dW_{s_1} dW_s + \cdots . \tag{3.75}$$

The integrands $\phi_s = \phi^\alpha(s), \phi_{ss_1} = \phi^{\alpha\alpha_1}(s, s_1), \phi_{ss_1s_2} = \phi^{\alpha\alpha_1\alpha_2}(s, s_1, s_2)$, and so on, appearing here are deterministic tensor–valued functions, where $s \geq s_1 \geq s_2 \geq \cdots$, and we note that any two terms of the expansion are orthogonal in expectation. Then for the expectation of the square of the random variable X_∞ we have

$$E\Big[X_\infty^2\Big] = \int_0^\infty \phi_s^2 ds + \int_0^\infty \int_{s_1=0}^s \phi_{ss_1}^2 ds_1 ds + \cdots \quad . \tag{3.76}$$

It should be evident by consideration of formula (3.66) that for each choice of X_∞ we obtain a specific interest rate model. In addition, the different models thus arising are nested in a natural way. If X_∞ can be chosen such that it only contains chaos elements up to order n, then we shall call the corresponding interest rate model an n^{th}–order chaos model. If X_∞ can be chosen such that it contains only an n^{th}–order term, we shall call the resulting interest rate model a 'pure' chaos model of order n. It should be clear that the n^{th}–order chaos models are contained as a subset of the m^{th}–order chaos models, for all $n < m$. Despite the relatively high level of abstraction in the overall framework, the inputs of such models are simply the deterministic functions ϕ_s, ϕ_{ss_1}, $\phi_{ss_1s_2}$ and so on. It follows that interest rate models can be classified according to their chaos structure, and indeed all positive interest HJM models based on a Brownian filtration can be systematically built up in this way.

3.11 Elements of Wiener chaos and its applications

Before we embark upon the analysis of specific interest rate models it will be helpful first if we review the basics of the Wiener chaos technique. This will also allow us to develop the notation further. The material discussed in this section is well established, and we refer the reader, e.g., to Nualart (1995), Øksendal (1997), or Teichmann (2002) for further details. The foundations of the chaos technique can be found in Wiener (1938) and Itô (1951). The applications of Wiener chaos to problems in finance were pioneered by Lacoste (1996).

Let H be a real Hilbert space with scalar product $\langle \cdot, \cdot \rangle$. Given an element $h \in H$, its norm will be denoted $\|h\|$. We introduce a field of random variables $W = \{W_h, h \in H\}$. We say that W is a Gaussian field if W is a Gaussian family of random variables with zero mean such that $E[W_g W_h] = \langle g, h \rangle$ for all $g, h \in H$. Under this definition the map $h \to W_h$ is a linear isometry of the space H onto a closed subspace of $L^2(\Omega, \mathcal{F}, P)$, which we denote by $\mathcal{H}_1$. It follows immediately that $W_{(ag+bh)} = aW_g + bW_h$ for any $a, b \in R$ and $g, h \in H$. The elements of $\mathcal{H}_1$ are zero–mean Gaussian random variables.

Next we introduce the Hermite polynomials $H_n(x)$, defined as usual by the formula

$$H_n(x) = \frac{1}{n!}(-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n}(e^{-\frac{1}{2}x^2}), \qquad n \geq 1, \tag{3.77}$$

and $H_0(x) = 1$. These polynomials play a fundamental role in the Wiener chaos expansion. The Hermite polynomials of degree one, two, three and four are $H_1(x) = x$, $H_2(x) = \frac{1}{2}(x^2 - 1)$, $H_3(x) = \frac{1}{6}(x^3 - 3x)$, and $H_4(x) = \frac{1}{24}(x^4 - 6x^2 + 3)$ respectively.

Let X and Y be random variables with a jointly Gaussian distribution such that $E[X] = E[Y] = 0$, and $E[X^2] = E[Y^2] = 1$. Then for all $n, m \geq 0$ we have

$$E[H_n(X)H_m(Y)] = \frac{1}{n!}\delta_{nm}\left(E\left[XY\right]\right)^n. \tag{3.78}$$

For each $n \geq 1$ we denote by $\mathcal{H}_n$ the linear subspace of $L^2(\Omega, \mathcal{F}, P)$ generated by the random variables $\{H_n(W_h), h \in H, \|h\| = 1\}$, with the convention that $\mathcal{H}_0$ denotes the constants. For $n = 1$, we recover the space $\mathcal{H}_1$ of zero mean Gaussian random variables. It should be evident from (3.78) that $\mathcal{H}_n$ and $\mathcal{H}_m$ are orthogonal for $n \neq m$. The subspace $\mathcal{H}_n$ is called the *Wiener chaos of order n*. If we denote by $\{\mathcal{G}\}$ the σ–field generated by the random variables $\{W_h, h \in H\}$, then the space $L^2(\Omega, \mathcal{G}, P)$ can be decomposed into the following infinite orthogonal sum of the subspaces $\mathcal{H}_n$:

$$L^2(\Omega, \mathcal{G}, P) = \oplus_{n=0}^{\infty} \mathcal{H}_n. \tag{3.79}$$

This fundamental decomposition of $L^2(\Omega, \mathcal{G}, P)$ leads to the representation of any element of this space by series of terms resulting from the orthogonal projection of the given element on to the various chaos subspaces.

Now let us reduce the generality of the underlying Hilbert space and consider the case $H = L^2(R_+, \mathcal{B}, \mu)$, where $\{\mathcal{B}\}$ denotes the Borel σ–algebra on R_+ and μ is the Lebesgue measure. In this case any element of the n^{th}–order Wiener chaos can be represented as an Itô integral of a square integrable function. To be more precise, let us consider the subspace Δ^n of R_+^n defined by

$$\Delta^n = \{(s, s_1, \ldots, s_{n-1}) \in R_+^n; 0 \leq s_{n-1} \leq \ldots \leq s_1 \leq s \leq \infty\}. \tag{3.80}$$

Let the function $\phi_n : R_+^n \to R$, be square integrable in the sense that

$$\int_0^{\infty} \int_0^{s} \cdots \int_0^{s_{n-2}} \phi_n^2(s, s_1, \ldots, s_{n-1}) ds_{n-1} \ldots ds_1 ds < \infty. \tag{3.81}$$

Then if we let $\{W_t\}$ denote a one–dimensional Brownian motion, we can verify that the random variable $I_n(\phi_n)$ defined by the multiple Itô integral

$$I_n(\phi_n) = \int_0^{\infty} \int_0^{s} \cdots \int_0^{s_{n-2}} \phi_n(s, s_1, \ldots, s_{n-1}) dW_{s_{n-1}} \ldots dW_{s_1} dW_s \tag{3.82}$$

is an element of the n^{th} Wiener chaos subspace $\mathcal{H}_n$. Indeed, the integral on the right hand side of the equation above is an Itô integral on Δ^n since the integrand is adapted and square integrable. Now let us write $\mathcal{F}_\infty^W$ for the σ–field generated by $\{W_t\}$ over the totality of the infinite time horizon. By combining expression (3.82) with the decomposition (3.79), one is led to the result that any square integrable random variable $X \in L^2(\Omega, \mathcal{F}_\infty^W, P)$ can be expressed as a chaos expansion according to the scheme

$$X = \sum_{n=0}^{\infty} I_n(\phi_n), \tag{3.83}$$

where the deterministic functions $\phi_n \in L^2(R_+^n)$ are uniquely determined by the random variable X (see, e.g., Revuz and Yor 2001).

It is a straightforward exercise to verify explicitly by use of the Itô isometry and the stochastic Fubini theorem (interchange of integration and expectation) that elements of distinct chaos spaces are orthogonal. For example, if $X \in \mathcal{H}_1$, and $Y \in \mathcal{H}_2$ we have

$$X = \int_0^\infty \phi(s) dW_s, \qquad \text{and} \qquad Y = \int_0^\infty \int_0^s \phi(s, s_1) dW_{s_1} dW_s, \tag{3.84}$$

for some choice of $\phi(s) \in L^2(R_+^1)$ and $\phi(s, s_1) \in L^2(R_+^2)$, and thus

$$\begin{aligned} E[XY] &= E\left[\int_0^\infty \phi(s) dW_s \int_0^\infty \int_0^s \phi(s, s_1) dW_{s_1} dW_s\right] \\ &= E\left[\int_0^\infty \int_0^s \phi(s)\phi(s, s_1) dW_{s_1} ds\right] \\ &= \int_0^\infty E\left[\int_0^s \phi(s)\phi(s, s_1) dW_{s_1}\right] ds \\ &= 0. \end{aligned} \tag{3.85}$$

On the other hand, if $A, B \in \mathcal{H}_2$ are two elements of the same chaos, e.g.,

$$A = \int_0^\infty \int_0^s \alpha(s, s_1) dW_{s_1} dW_s, \qquad B = \int_0^\infty \int_0^s \beta(s, s_1) dW_{s_1} dW_s, \tag{3.86}$$

then their inner product is given by

$$\begin{aligned} E[AB] &= E\left[\int_0^\infty \int_0^s \alpha(s, s_1) dW_{s_1} dW_s \int_0^\infty \int_0^s \beta(s, s_1) dW_{s_1} dW_s\right] \\ &= E\left[\int_0^\infty \left(\int_0^s \alpha(s, s_1) dW_{s_1} \int_0^s \beta(s, s_1) dW_{s_1}\right) ds\right] \end{aligned}$$

$$= \int_0^\infty E\left[\int_0^s \alpha(s,s_1)dW_{s_1}\int_0^s \beta(s,s_1)dW_{s_1}\right]ds$$
$$= \int_0^\infty \int_0^s \alpha(s,s_1)\beta(s,s_1)ds_1 ds. \tag{3.87}$$

Thus the random variables A and B are orthogonal in $\mathcal{H}_2$ if and only if the corresponding elements of $L^2(R_+^2)$ are orthogonal.

Another useful result arises in the case for which $\phi_n(t_1, t_2, \ldots, t_n)$ is 'factorisable' in the special form

$$\phi_n(s, s_1, \ldots, s_{n-1}) = h(s)h(s_1)\ldots h(s_{n-1}), \tag{3.88}$$

for some element $h(t) \in L^2(R_+^1)$ with unit norm. Then for this choice of ϕ_n we have the relation $I_n(\phi_n) = H_n(W_h)$, where $H_n(W_h)$ is the n^{th} Hermite polynomial formed from the unit–norm Gaussian random variable W_h defined by

$$W_h = \int_0^\infty h(s)dW_s, \qquad \int_0^\infty h^2(s)ds = 1. \tag{3.89}$$

We note, in particular, that

$$\exp\left[\alpha W_h - \frac{1}{2}\alpha^2\right] = \sum_{n=0}^{\infty} \alpha^n H_n(W_h). \tag{3.90}$$

The formulae presented in this section apply in the case of the Wiener chaos based on a standard one–dimensional Brownian motion. The extension to the general case of a multidimensional Brownian motion is straightforward, and consists of replacing the deterministic coefficients ϕ_s, ϕ_{ss_1}, $\phi_{ss_1s_2}$, etc., with appropriate tensorial expressions of the form $\phi^{\alpha}(s)$, $\phi^{\alpha\alpha_1}(s,s_1)$, $\phi^{\alpha\alpha_1\alpha_2}(s,s_1,s_2)$, and so on.

Chapter 4

Wiener chaos models and the pricing of interest rate derivatives

We now consider the question of pricing interest rate derivatives by use of the chaotic framework introduced in the previous chapter.

4.1 First chaos models

The first Wiener chaos offers the simplest application of the method and gives rise to a deterministic interest rate model. One should remember that the majority of the applications of interest rate theory start from the deterministic case, so this case should not be regarded as 'trivial'. Indeed, the chaos framework offers new insights into the relation between deterministic models and their stochastic generalisations. It is interesting to note in this connection that even in the case of a deterministic interest rate model there is still a random variable underpinning the dynamics. For simplicity we shall assume here that the dimension of the Brownian motion is one. In the case of a first chaos model we then write

$$X_\infty = \int_0^\infty \phi_s dW_s, \tag{4.1}$$

where ϕ_s is a deterministic function of one variable. A straightforward calculation by use of the Itô isometry confirms that the corresponding expression for the pricing kernel is given by

$$V_t = \int_t^\infty \phi_s^2 ds. \tag{4.2}$$

Indeed, we have

$$E_t[X_\infty] = \int_0^t \phi_s dW_s, \tag{4.3}$$

and thus

$$X_\infty - E_t[X_\infty] = \int_t^\infty \phi_s dW_s. \tag{4.4}$$

Then we find that $E_t[(X_\infty - E_t[X_\infty])^2]$ is given by (4.2). This expression for $\{V_t\}$ is clearly a positive supermartingale that tends to zero in expectation, and it is evident that the interest rate model that arises is deterministic. The corresponding expression for the discount bonds is

$$P_{tT} = \frac{\int_T^\infty \phi_s^2 ds}{\int_t^\infty \phi_s^2 ds}. \tag{4.5}$$

Thus, the first chaos is sufficient to characterise a deterministic interest rate structure. In other words, we can identify the space of positive interest yield curves with the first chaos.

4.2 Second chaos models

The second chaos models are the simplest models that introduce stochasticity. In a single–factor second chaos model the random variable X_∞ can be represented in the form (3.62), with the adapted process $\{\sigma_s\}$ given by

$$\sigma_s = \phi_s + \int_0^s \phi_{ss_1} dW_{s_1}. \tag{4.6}$$

Here ϕ_s, $(0 \leq s < \infty)$ is a deterministic function of one variable, and ϕ_{ss_1}, $(0 \leq s \leq s_1 < \infty)$ is a deterministic function of two variables. The second chaos representation for X_∞ is then given by

$$X_\infty = \int_0^\infty \phi_s dW_s + \int_0^\infty \int_0^s \phi_{ss_1} dW_{s_1} dW_s. \tag{4.7}$$

In the case of a second chaos model we observe that *the deterministic coefficients ϕ_s and ϕ_{ss_1} supply just enough freedom to allow for calibration to the initial yield curve and a complete set of caplet prices for all tenors and maturities.*

It is a straightforward exercise to show as a consequence of equation (3.66) that we are then led to the following expression for the pricing kernel

$$V_t = \int_t^\infty \left(\phi_s + \int_0^t \phi_{ss_1} dW_{s_1} \right)^2 ds + \int_t^\infty \int_t^s \phi_{ss_1}^2 ds_1 ds. \tag{4.8}$$

The derivation of formula (4.8) can be established most directly if we write

$$V_t = \int_t^\infty M_{ts} ds, \tag{4.9}$$

where the positive martingale family $\{M_{ts}\}$ is defined for $0 \leq t \leq s \leq \infty$ by the relation

$$M_{ts} = E_t \left[\sigma_s^2 \right]. \tag{4.10}$$

The fact that the process $\{V_t\}$ can be represented in this way follows as a consequence of the first line of (3.64). Then a short calculation making use of equation (4.6) and the conditional Itô isometry gives

$$M_{ts} = \left(\phi_s + \int_0^t \phi_{ss_1} dW_{s_1} \right)^2 + \int_t^s \phi_{ss_1}^2 ds_1. \tag{4.11}$$

To check that the expression appearing on the right hand side of (4.11) is indeed a martingale we note that

$$M_{ts} = R_{ts}^2 - Q_{ts} + Q_{ss}, \tag{4.12}$$

where, for each value of s, the process $\{R_{ts}\}$ is the martingale

$$R_{ts} = \phi_s + \int_0^t \phi_{ss_1} dW_{s_1}, \tag{4.13}$$

and $\{Q_{ts}\}$ is the associated quadratic variation:

$$Q_{ts} = \int_0^t \phi_{ss_1}^2 ds_1. \tag{4.14}$$

If $\{R_{ts}\}$ is a martingale and $\{Q_{ts}\}$ is its quadratic variation, then $\{R_{ts}^2 - Q_{ts}\}$ is also a martingale, and hence so is $\{M_{ts}\}$ since $\{Q_{ss}\}$ is deterministic and independent of t. On the other hand $\{Q_{ss}\}$ is just the extra term required to ensure $\{M_{ts}\}$ is positive for all $0 \leq t \leq s \leq \infty$, as is clear from expression (4.11). The discount bond system can then be put into the Flesaker–Hughston form

$$P_{tT} = \frac{\int_T^\infty M_{ts} ds}{\int_t^\infty M_{ts} ds}, \tag{4.15}$$

and the initial term structure that corresponds to this system is given by

$$P_{0T} = \frac{\int_T^\infty M_{0s} ds}{\int_0^\infty M_{0s} ds}. \tag{4.16}$$

More explicitly, we have

$$M_{0s} = \phi_s^2 + \int_0^s \phi_{ss_1}^2 ds_1, \tag{4.17}$$

and hence:

$$P_{0T} = \frac{\int_T^\infty \left(\phi_s^2 + \int_0^s \phi_{ss_1}^2 ds_1\right) ds}{\int_0^\infty \left(\phi_s^2 + \int_0^s \phi_{ss_1}^2 ds_1\right) ds}. \tag{4.18}$$

By an overall adjustment of the scale of X_∞ we can set the denominator in (4.18) to unity. With this choice of normalisation the corresponding term structure density is given by

$$\rho(T) = M_{0T}. \tag{4.19}$$

4.3 Second chaos bond volatility and market price of risk

We now proceed to calculate expressions for the discount bond volatility and the market price of risk arising in the case of a general second chaos model.

Making use of the Itô quotient identity

$$\frac{d\,(A_t/B_t)}{(A_t/B_t)} = \frac{dA_t}{A_t} - \frac{dB_t}{B_t} + \frac{(dB_t)^2}{B_t^2} - \frac{dA_t dB_t}{A_t B_t}, \tag{4.20}$$

we deduce that the discount bond volatility is given by

$$\Omega_{tT} \;=\; \frac{\int_T^\infty U_{ts}ds}{\int_T^\infty M_{ts}ds} - \frac{\int_t^\infty U_{ts}ds}{\int_t^\infty M_{ts}ds}, \tag{4.21}$$

and that the market risk premium vector is given by

$$\lambda_t = -\frac{\int_t^\infty U_{ts}ds}{\int_t^\infty M_{ts}ds}. \tag{4.22}$$

Here for convenience we have introduced the vector–valued process $\{U_{ts}\}$ defined by

$$U_{ts} = 2R_{ts}\phi_{st}. \tag{4.23}$$

We note that the constraint

$$\Omega_{TT} = 0, \tag{4.24}$$

is automatically satisfied.

The instantaneous forward rate process $\{f_{tT}\}$ can be calculated by use of the formula

$$f_{tT} = -\frac{\partial}{\partial T}(\ln P_{tT}), \tag{4.25}$$

and we find

$$f_{tT} = \frac{M_{tT}}{\int_T^\infty M_{ts}ds}. \tag{4.26}$$

The short rate process is given analogously by the formula

$$r_t = \frac{M_{tt}}{\int_t^\infty M_{ts}ds}, \tag{4.27}$$

which is equivalent to the relation

$$\sigma_t^2 = r_t V_t. \tag{4.28}$$

4.4 Option pricing in general second chaos models

At first glance, the expressions related to the second chaos might look complicated. However the only exogenously specified ingredients are the deterministic functions ϕ_s and ϕ_{ss_1}. In fact, all the formulae above can be expressed in terms of the underlying Gaussian random variables $\{R_{ts}\}$. Encouraged by this fact, let us examine the problem of option pricing in a second chaos model. We observe that for fixed values of t and s the random variable M_{ts} defined by (4.11) is given by the square of a Gaussian random variable, plus a constant. Therefore, for fixed t and T the random variable

$$Z_{tT} = \int_T^{\infty} M_{ts} ds, \tag{4.29}$$

can be understood as the integral of a parametric family of squared Gaussian random variables, plus a constant. The next step is to define the joint distribution function of the random variables Z_{tT_1}, and Z_{tT_2} by

$$F_{tT_1T_2}(x, y) = \text{Prob}\left[Z_{tT_1} \leq x \text{ and } Z_{tT_2} \leq y\right]. \tag{4.30}$$

We denote the corresponding joint density function by $f_{tT_1T_2}(x, y)$. Now the payoff for a call option that expires at time t and is written on a T–maturity discount bond is

$$H_t = (P_{tT} - K)^+, \tag{4.31}$$

for some strike K. Therefore, according to (3.40) the price of this instrument is

$$H_0 = E\Big[V_t\Big(P_{tT} - K\Big)^+\Big]. \tag{4.32}$$

By virtue of (4.15) this is evidently equivalent to

$$H_0 = E\Big[\Big(Z_{tT} - KZ_{tt}\Big)^+\Big], \tag{4.33}$$

which can be written in terms of the density function $f(x, y)$ in the form

$$H_0 = \int_0^\infty \int_0^\infty f(x, y)\Big(x - Ky\Big)^+ dxdy. \tag{4.34}$$

Analogous formulae can be derived for other types of options.

4.5 Factorisable second chaos models

A considerable simplification can be achieved when the second chaos coefficient ϕ_{ss_1} separates, that is to say, when ϕ_{ss_1} can be written as a finite sum of products of functions of one variable. In this situation we obtain a model characterised by a finite set of state variables. We shall examine in some detail the case where there is a single such term, and set $\phi_s = \alpha_s$ and $\phi_{ss_1} = \beta_s\gamma_{s_1}$, where α_s, β_s and γ_{s_1} are deterministic functions of one variable. The resulting 'factorisable' second chaos model then depends on a single state variable, and is completely tractable in the sense that it leads to closed–form expressions both for bond prices and various types of options on bond prices, which we discuss at greater length below.

First we observe that in the factorisable case we have

$$\phi_s + \int_0^t \phi_{ss_1} dW_{s_1} = \alpha_s + \beta_s R_t, \tag{4.35}$$

where the Gaussian martingale $\{R_t\}$ is defined by

$$R_t = \int_0^t \gamma_{s_1} dW_{s_1}. \tag{4.36}$$

At any given time t, the random variable R_t is the sole state variable that characterises the interest rate system in this model. If we define the corresponding quadratic variation process $\{Q_t\}$ by

$$Q_t = \int_0^t \gamma_s^2 ds, \tag{4.37}$$

then it follows that the process $\{R_t^2 - Q_t\}$ is also a martingale, and the positive martingale family $\{M_{ts}\}$ defined by (4.11) reduces to the expression

$$M_{ts} = \alpha_s^2 + \beta_s^2 Q_s + 2\alpha_s\beta_s R_t + \beta_s^2\left(R_t^2 - Q_t\right). \tag{4.38}$$

Clearly, $\{Q_s\} \geq \{Q_t\}$ for all $s \geq t$, so $\{M_{ts}\} > 0$ for all values of $\{R_t\}$. For the integral of $\{M_{ts}\}$ we can write

$$\int_T^\infty M_{ts} ds = A_T + B_T R_t + C_T \left(R_t^2 - Q_t\right), \tag{4.39}$$

where for convenience in what follows we define the following processes:

$$\begin{aligned} A_t &= \int_t^\infty \left(\alpha_s^2 + \beta_s^2 Q_s\right) ds, \\ B_t &= 2\int_t^\infty \alpha_s \beta_s ds, \\ C_t &= \int_t^\infty \beta_s^2 ds. \end{aligned} \tag{4.40}$$

Setting $T = t$ in (4.39) we see that the pricing kernel is given by

$$V_t = A_t + B_t R_t + C_t \left(R_t^2 - Q_t\right), \tag{4.41}$$

and thus that the discount bond price can be written as the ratio of a pair of quadratic polynomials in the state variable R_t :

$$P_{tT} = \frac{A_T + B_T R_t + C_T \left(R_t^2 - Q_t\right)}{A_t + B_t R_t + C_t \left(R_t^2 - Q_t\right)}. \tag{4.42}$$

Given these expressions, it is then a straightforward exercise to work out formulae for the bond volatility, the market price of risk, the instantaneous forward rates, and the short rate, all of which depend upon R_t. Since $\{R_t\}$ is a Gaussian martingale, it is in principle straightforward to simulate the dynamical trajectories of these quantities. In particular, we have:

$$\Omega_{tT} = \frac{\gamma_t B_T + 2\gamma_t C_T R_t}{A_T + B_T R_t + C_T(R_t^2 - Q_t)} - \frac{\gamma_t B_t + 2\gamma_t C_t R_t}{A_t + B_t R_t + C_t(R_t^2 - Q_t)}, \tag{4.43}$$

$$\lambda_t = -\frac{\gamma_t B_t + 2\gamma_t C_t R_t}{A_t + B_t R_t + C_t(R_t^2 - Q_t)}, \tag{4.44}$$

$$f_{tT} = \frac{\alpha_T^2 + \beta_T^2 Q_T + 2\alpha_T \beta_T R_t + \beta_T^2 (R_t^2 - Q_t)}{A_T + B_T R_t + C_T(R_t^2 - Q_t)}, \tag{4.45}$$

$$r_t = \frac{\alpha_t^2 + \beta_t^2 Q_t + 2\alpha_t \beta_t R_t + \beta_t^2 (R_t^2 - Q_t)}{A_t + B_t R_t + C_t(R_t^2 - Q_t)}. \tag{4.46}$$

4.6 Option pricing in factorisable second chaos models

Now let us look at the problem of the evaluation of certain types of options in the factorisable case. The present value H_0 of a European–style call option with strike K exercisable at time t on a discount bond with maturity T is given by (4.32). According to (4.41) and (4.42), we have

$$\begin{aligned} V_t\left(P_{tT}-K\right) &= \left(A_T-KA_t\right)-\left(C_T-KC_t\right)Q_t \\ &+ \left(B_T-KB_t\right)R_t+\left(C_T-KC_t\right)R_t^2. \end{aligned} \tag{4.47}$$

To proceed let us therefore now fix t, T and K, and introduce a standard normally distributed random variable

$$Z=\frac{R_t}{\sqrt{Q_t}}. \tag{4.48}$$

Then (4.47) can be written in the form

$$V_t(P_{tT}-K)=A+BZ+CZ^2, \tag{4.49}$$

where the quantities A, B and C are defined by:

$$\begin{aligned} A &= (A_T-KA_t)-(C_T-KC_t)Q_t, \\ B &= (B_T-KB_t)Q_t^{1/2}, \\ C &= (C_T-KC_t)Q_t. \end{aligned} \tag{4.50}$$

Therefore if we construct the polynomial

$$\mathcal{P}(z)=A+Bz+Cz^2, \tag{4.51}$$

we see that the value of the call option is given by

$$H_0=\frac{1}{\sqrt{2\pi}}\int_{\mathcal{P}(z)\geq 0}\mathcal{P}(z)e^{-\frac{1}{2}z^2}dz, \tag{4.52}$$

which by an analysis of the roots of $\mathcal{P}(z)$ can be reduced to a simple explicit expression involving the normal distribution function and its density. Analogous formulae can then be deduced for various other types of options, as we shall indicate shortly.

Let us proceed then case by case to examine the behaviour of the polynomial $\mathcal{P}(z)$ more closely. First we distinguish the cases $C = 0$ and $C \neq 0$. If $C = 0$ then $\mathcal{P}(z)$ is linear, and for the value of the call option we obtain

$$H_0 = AN(-z_0) + B\rho(z_0), \tag{4.53}$$

when $B > 0$, and

$$H_0 = AN(z_0) - B\rho(z_0), \tag{4.54}$$

when $B < 0$. Here

$$z_0 = -\frac{A}{B}, \tag{4.55}$$

is the single root of $\mathcal{P}(z)$, $N(z)$ is the standard normal distribution function, and $\rho(z)$ is the standard normal density function.

If $C \neq 0$, then we need to consider the sign of the discriminant

$$\Delta = B^2 - 4AC. \tag{4.56}$$

If $\Delta \leq 0$ then for $C > 0$ the option is guaranteed to expire in the money, and we have

$$H_0 = P_{0T} - KP_{0t}. \tag{4.57}$$

If $C < 0$ then the option will expire out of the money and $H_0 = 0$. If $\Delta > 0$ then, again, we have to consider the cases $C > 0$ and $C < 0$. Let us write

$$z_1 = \frac{-B - \sqrt{\Delta}}{2C}, \qquad z_2 = \frac{-B + \sqrt{\Delta}}{2C}, \tag{4.58}$$

for the roots of $\mathcal{P}(z)$. Then if $C > 0$ we obtain

$$\begin{aligned} H_0 &= (P_{0T} - KP_{0t})\left(N(z_1) + N(-z_2)\right) \\ &- \frac{1}{2}\left(B - \sqrt{\Delta}\right)\rho(z_1) + \frac{1}{2}\left(B + \sqrt{\Delta}\right)\rho(z_2), \end{aligned} \tag{4.59}$$

and if $C < 0$ we obtain

$$\begin{aligned} H_0 &= (P_{0T} - KP_{0t})\,(N(z_1) - N(z_2)) \\ &- \frac{1}{2}\left(B - \sqrt{\Delta}\right)\rho(z_1) + \frac{1}{2}\left(B + \sqrt{\Delta}\right)\rho(z_2). \end{aligned} \tag{4.60}$$

Thus we see that in the factorisable second–chaos framework the pricing of options on discount bonds is completely tractable. More generally, the value of an option on any predesignated set of deterministic cash–flows is also tractable, for example an option on a coupon bond. To obtain the above formulae, we have set $A_0 = 1$. This can be achieved without loss of generality by changing the scale of X_∞.

4.7 Swaption pricing in factorisable second chaos models

Now we shall demonstrate that in the factorisable second–chaos framework we can also derive explicit results for a swaption that pays $(S_{tn} - K)^+$ at a series of future dates T_i, for some strike K, where $i = 1, \cdots, n$, and S_{tn} is the swap rate

$$S_{tn} = \frac{1 - P_{tT_n}}{\sum_{i=1}^{n} P_{tT_i}}. \tag{4.61}$$

The effective payoff at expiry t is therefore equal to

$$H_t = \left(1 - P_{tT_n} - K\sum_{i=1}^{n} P_{tT_i}\right)^+, \tag{4.62}$$

and the price for this instrument at present is:

$$H_0 = E\left[V_t\left(1 - P_{tT_n} - K\sum_{i=1}^{n} P_{tT_i}\right)^+\right]. \tag{4.63}$$

The analysis is quite similar to that of the bond option case. In the case of a swaption we define the quantities

$$A^* \;=\; \left(A_t - A_{T_n} - K\sum_{i=1}^{n} A_{T_i}\right) - \left(C_t - C_{T_n} - K\sum_{i=1}^{n} C_{T_i}\right) Q_t,$$

$$
\begin{aligned}
B^* &= \left(B_t - B_{T_n} - K \sum_{i=1}^{n} B_{T_i} \right) Q_t^{1/2}, \\
C^* &= \left(C_t - C_{T_n} - K \sum_{i=1}^{n} C_{T_i} \right) Q_t,
\end{aligned} \tag{4.64}
$$

for fixed t and T_i. The value of the swaption is then given by (4.52), where in the present case the polynomial $\mathcal{P}(z)$ is given by

$$
\mathcal{P}(z) = A^* + B^* z + C^* z^2. \tag{4.65}
$$

When $C^* = 0$ we have:

$$
H_0^* = A^* N(-z_0^*) + B^* \rho(z_0^*), \tag{4.66}
$$

for $B^* > 0$, and

$$
H_0^* = A^* N(z_0^*) - B^* \rho(z_0^*), \tag{4.67}
$$

for $B^* < 0$. Here $z_0^* = -A^*/B^*$. When $C^* \neq 0$ then we have to consider the discriminant:

$$
\Delta^* = B^{*2} - 4A^* C^*. \tag{4.68}
$$

If $\Delta^* \leq 0$ then for $C^* > 0$ the contract is guaranteed to pay off and the value at present is:

$$
H_0^* = P_{0t} - P_{0T_n} - K \sum_{i=1}^{n} P_{0T_i}. \tag{4.69}
$$

On the other hand if $\Delta^* \leq 0$ and $C^* < 0$ then the contract will expire worthless and $H_0^* = 0$. Finally, when $\Delta^* > 0$ we define the two roots of $\mathcal{P}(z)$ by

$$
z_1^* = \frac{-B^* - \sqrt{\Delta^*}}{2C^*}, \qquad z_2^* = \frac{-B^* + \sqrt{\Delta}}{2C^*}. \tag{4.70}
$$

The value of the swaption contract is then given by

$$
\begin{aligned}
H_0^* &= \left(P_{0t} - P_{0T_n} - K \sum_{i=1}^{n} P_{0T_i} \right) \left(N(z_1^*) + N(-z_2^*) \right) \\
&- \frac{1}{2} \left(B^* - \sqrt{\Delta^*} \right) \rho(z_1^*) + \frac{1}{2} \left(B^* + \sqrt{\Delta^*} \right) \rho(z_2^*),
\end{aligned} \tag{4.71}
$$

when $C^* > 0$; whereas if $C^* < 0$ we get

$$\begin{aligned} H_0^* &= \left(P_{0t} - P_{0T_n} - K \sum_{i=1}^{n} P_{0T_i} \right) (N(z_1^*) - N(z_2^*)) \\ &- \frac{1}{2} \left(B^* - \sqrt{\Delta^*} \right) \rho(z_1^*) + \frac{1}{2} \left(B^* + \sqrt{\Delta^*} \right) \rho(z_2^*). \end{aligned} \tag{4.72}$$

It is a remarkable feature of the factorisable second chaos models that they admit tractable closed–form expressions for both options and swaptions.

4.8 Third chaos models

A considerably richer structure emerges when we address the general third chaos models, for which the first three expansion coefficients ϕ_s, ϕ_{ss_1}, and, $\phi_{ss_1s_2}$ are non–vanishing. In this case the positive family of martingales $\{M_{ts}\}$ takes the form

$$\begin{aligned} M_{ts} &= \left(\phi_s + \int_0^t \phi_{ss_1} dW_{s_1} + \int_0^t \int_0^{s_1} \phi_{ss_1s_2} dW_{s_2} dW_{s_1} \right)^2 \\ &+ \int_t^s \left(\phi_{ss_1} + \int_0^t \phi_{ss_1s_2} dW_{s_2} \right)^2 ds_1 + \int_t^s \int_t^{s_1} \phi_{ss_1s_2}^2 ds_2 ds_1. \end{aligned} \tag{4.73}$$

To see this, consider again the definition (3.62). The adapted process $\{\sigma_t\}$ now takes the form:

$$\sigma_s = \phi_s + \int_0^s \phi_{ss_1} dW_{s_1} + \int_0^s \int_0^{s_1} \phi_{ss_1s_2} dW_{s_2} dW_{s_1}. \tag{4.74}$$

Now recall that

$$M_{ts} = E_t[\sigma_s^2]. \tag{4.75}$$

By combining these two expression and by use of Itô isometry we then recover (4.73). The pricing kernel is then given as usual by

$$V_t = \int_t^\infty M_{ts} ds, \tag{4.76}$$

and the discount bond system is determined by use of equation (4.15).

The analysis above is simplified significantly when we consider the case of a third factorisable chaos model. That is we put $\phi_s = \alpha_s, \phi_{ss_1} = \beta_s \gamma_{s_1}$ and $\phi_{ss_1 s_2} = \delta_s \epsilon_{s_1} \zeta_{s_2}$. We define the random variables

$$R_t = \int_0^t \gamma_s dW_s, \qquad L_t = \int_0^t \zeta_s dW_s, \qquad N_t = \int_0^t \epsilon_s L_s dW_s, \tag{4.77}$$

as well as the following deterministic functions:

$$\begin{aligned} Q_t = \int_0^t \gamma_s^2 ds, \qquad G_t = \int_0^t \zeta_s^2 ds, \qquad E_t = \int_0^t \epsilon_s^2 ds, \\ H_t = \int_0^t \epsilon_s^2 G_s ds, \qquad \Theta_t = \int_0^t \gamma_s \epsilon_s ds. \end{aligned} \tag{4.78}$$

In addition we introduce the random variable

$$X_{ts} = \beta_s R_t + \delta_s N_t, \tag{4.79}$$

and the corresponding deterministic function

$$Y_{ts} = \beta_s^2 Q_t + \delta_s^2 H_t. \tag{4.80}$$

The definitions above serve as a convenient notation that makes the expressions to follow both compact and suggestive. In particular, the martingale family $\{M_{ts}\}$ takes the form

$$\begin{aligned} M_{ts} &= X_{ts}^2 - Y_{ts} + 2\alpha_s X_{ts} + \alpha_s^2 + Y_{ss} \\ &+ 2\beta_s \delta_s (\Theta_s - \Theta_t) L_t + \delta_s^2 (E_s - E_t) \left(L_t^2 - G_t\right). \end{aligned} \tag{4.81}$$

It is then a straightforward exercise to derive the pricing kernel for this model

$$\begin{aligned} V_t &= \int_t^\infty \left(X_{ts}^2 - Y_{ts}\right) ds + 2\int_t^\infty \alpha_s X_{ts} ds \\ &+ \int_t^\infty \left(\alpha_s^2 + Y_{ss}\right) ds + \Lambda_t^1 L_t + \Lambda_t^2 \left(L_t^2 - G_t\right), \end{aligned} \tag{4.82}$$

where we further define

$$\Lambda_t^1 = 2\int_t^\infty \beta_s \delta_s (\Theta_s - \Theta_t) ds, \tag{4.83}$$

and

$$\Lambda_t^2 = \int_t^\infty \delta_s^2 (E_s - E_t) ds. \tag{4.84}$$

Note that the representation for the pricing kernel in the case of the second factorisable chaos immediately pops out from the above formula once we consider for example $\delta = 0$. Returning to the discussion about the third chaos, we note that the discount bond system is defined accordingly via (4.15). To this end we remark that the factorisable third chaos models presented here admit a calibration to compound interest rate options.

4.9 Finite chaos models

Any chaos expansion that consists of a finite number of terms results in a finite chaos interest rate model. To this end we provide a useful formula for the associated family of positive martingales for a finite chaos model of order n. This is a recursive relation, and it can be shown that the previous examples for the first, second and third chaos are special cases. This takes the form:

$$\begin{aligned} M_{ts} &= \sum_{l=2}^{n+1} \Bigg[\int_t^s \int_t^{s_1} \cdots \int_t^{s_{l-3}} \Bigg(\phi_{ss_1 \cdots s_r} \mathbf{1}_{\{l=r+2\}} \\ &+ \sum_{m=l}^{n} \int_0^t \int_0^{s_1} \cdots \int_0^{s_{m-l}} \phi_{ss_1 \cdots s_{m-1}} dW_{s_{m-1}} \cdots dW_{s_{l-1}} \Bigg)^2 ds_{l-2} \cdots ds_1 \Bigg], \; n \geq 2 . \end{aligned} \tag{4.85}$$

The standard convention concerning the above formula is that we do not perform the integration when the subscripts in the integral limits are negative, which happens only for $l = 2$ and $m < l$, and when the subscript is zero we only perform the first integration. Thus, for example for $n = 2$ we recover expression (4.11):

$$M_{ts} = \left(\phi_s + \int_0^t \phi_{ss_1} dW_{s_1} \right)^2 + \int_t^s \phi_{ss_1}^2 ds_1, \tag{4.86}$$

where as for $n = 3$ we recover expression (4.73):

$$\begin{aligned}
M_{ts} &= \left(\phi_s + \int_0^t \phi_{ss_1} dW_{s_1} + \int_0^t \int_0^{s_1} \phi_{ss_1s_2} dW_{s_2} dW_{s_1}\right)^2 \\
&+ \int_t^s \left(\phi_{ss_1} + \int_0^t \phi_{ss_1s_2} dW_{s_2}\right)^2 ds_1 + \int_t^s \int_t^{s_1} \phi^2_{ss_1s_2} ds_2 ds_1. \quad (4.87)
\end{aligned}$$

When $n = 4$ the recursive relation (4.85) transforms to the family of positive martingales associated to the general fourth chaos theory:

$$\begin{aligned}
M_{ts} &= \left(\phi_s + \int_0^t \phi_{ss_1} dW_{s_1} + \int_0^t \int_0^{s_1} \phi_{ss_1s_2} dW_{s_2} dW_{s_1}\right. \\
&+ \left.\int_0^t \int_0^{s_1} \int_0^{s_2} \phi_{ss_1s_2s_3} dW_{s_3} dW_{s_2} dW_{s_1}\right)^2 \\
&+ \int_t^s \left(\phi_{ss_1} + \int_0^t \phi_{ss_1s_2} dW_{s_2} + \int_0^t \int_0^{s_1} \phi_{ss_1s_2s_3} dW_{s_3} dW_{s_2}\right)^2 ds_1 \\
&+ \int_t^s \int_t^{s_1} \left(\phi_{ss_1s_2} + \int_0^t \phi_{ss_1s_2s_3} dW_{s_3}\right)^2 ds_2 d_{s_1} \\
&+ \int_t^s \int_t^{s_1} \int_t^{s_2} \phi^2_{ss_1s_2s_3} ds_3 ds_2 ds_1 \,. \quad (4.88)
\end{aligned}$$

4.10 Pure chaos models

When only the n–th order term is non–vanishing we call the corresponding model *pure n–chaos*. In this situation the only exogenously specified component is a deterministic square–integrable function of n variables. Expression (4.85) then simplifies to

$$M_{ts} = \sum_{l=2}^{n+1} \int_t^s \int_t^{s_1} \cdots \int_t^{s_{l-3}} I_l^2 \quad ds_{l-2} \cdots ds_1, \quad (4.89)$$

where we define

$$I_l = \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-l}} \phi_{ss_1 \cdots s_{n-1}} dW_{s_{n-1}} \cdots dW_{s_{l-1}}. \quad (4.90)$$

Accordingly, in the pure *factorisable* case we have

$$M_{ts} = \sum_{l=2}^{n+1} \int_t^s \int_t^{s_1} \cdots \int_t^{s_{l-3}} J_l^2 \quad d_{s_{l-2}} \cdots ds_1, \quad (4.91)$$

where

$$J_l = \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-l}} \phi_s^{(0)} \phi_{s_1}^{(1)} \cdots \phi_{s_{n-1}}^{(n-1)} dW_{s_{n-1}} \cdots dW_{s_{l-1}}. \tag{4.92}$$

Things can be simplified further in what we shall call the *totally degenerate* case. This is when we have that

$$\phi^{(0)} = \phi^{(1)} = \cdots = \phi^{(n-1)} = \phi. \tag{4.93}$$

Then again we will have the relation

$$M_{ts} = \sum_{l=2}^{n+1} \int_t^s \int_t^{s_1} \cdots \int_t^{s_{l-3}} K_l^2 \quad d_{s_{l-2}} \cdots ds_1, \tag{4.94}$$

where now

$$K_l = \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-l}} \phi_s \phi_{s_1} \cdots \phi_{s_{n-1}} dW_{s_{n-1}} \cdots dW_{s_{l-1}}. \tag{4.95}$$

The expression above can be written in the semi–analytical form by means of the Hermite polynomials as

$$M_{ts} = E_t \left[H_{n-1}^2 (\widetilde{W}_\phi) \right], \qquad \widetilde{W}_\phi = \int_0^s \phi_u dW_u. \tag{4.96}$$

As a result of the above considerations, we conclude with an analytical formula for the discount bond system in the case of a totally degenerate n pure chaos model, in terms of Hermite polynomials:

$$P_{tT} = \frac{\sum_{l=2}^{n+1} H_{n-l+1}^2 (W_\phi)\, A_{Tl}}{\sum_{l=2}^{n+1} H_{n-l+1}^2 (W_\phi)\, A_{tl}}, \tag{4.97}$$

where

$$A_{Tl} = \int_T^\infty \int_t^s \int_t^{s_1} \cdots \int_t^{s_{l-3}} \phi_s^2 \phi_{s_1}^2 \cdots \phi_{s_{l-2}}^2 ds_{l-2} \cdots ds_1 ds, \tag{4.98}$$

and we now define $W_\phi = \int_0^t \phi_s dW_s$. The convention for the calculation of the above integral is slightly different from before; for the calculation of (4.98) we consider the first integration only when $l = 2$ and the first two integrations when $l = 3$. We also make use of the fact that the Hermite polynomial of degree zero is by definition unity. One can use this expression to recover the first and the degenerate pure factorisable second chaos discount bond systems. It is interesting to note that the first chaos is deterministic, as was shown earlier, but all other models will necessarily be stochastic. However, the superposition of *all* such terms, will again give a deterministic discount bond system. It should be evident that the expression above can be very useful when one faces the task of deriving explicit results for finite chaos models of some higher order. Based on expression (4.85), we can derive results for both the pricing kernel process and the discount bond system, without having to calculate a conditional variance each time, which can be difficult and time–consuming. To demonstrate the results above we provide as an example the third chaos case where we consider the factorisable framework. Then expression (4.91) becomes

$$M_{ts} = \delta_s^2 \left(N_t^2 + H_s - H_t + \left(L_t^2 - G_t\right)(E_s - E_t)\right), \tag{4.99}$$

which leads to the following expression for the discount bonds

$$P_{tT} = \frac{N_t^2 \int_T^\infty \delta_s^2 ds + \int_T^\infty \delta_s^2 (H_s - H_t)\, ds + (L_t^2 - G_t) \int_T^\infty \delta_s^2 (E_s - E_t)\, ds}{N_t^2 \int_t^\infty \delta_s^2 ds + \int_t^\infty \delta_s^2 (H_s - H_t)\, ds + (L_t^2 - G_t) \int_t^\infty \delta_s^2 (E_s - E_t)\, ds}. \tag{4.100}$$

4.11 Chaos and coherence

Following the discussion so far, one is tempted to ask what happens if we consider all the terms in the Wiener chaos expansion for the generator of an interest rate model. It turns out that in the case where the relevant chaos coefficients factorise and moreover they are all the same function, some very interesting results of explicit nature arise. We give here a description of these ideas. What follows for the rest of this section relies on Brody and Hughston (2004).

If each coefficient $\phi_{s_1 s_2 \cdots s_n}$ in the chaos expansion factorises into a product of n functions of the variables $s_1, s_2, \cdots, s_n$, then, as mentioned before, the resulting term structure simplifies enormously, and in some cases we can create models that are analytically tractable. We will examine the simplest totally degenerate factorisable model that involves *all* terms in the chaos expansion. For simplicity we consider the case when $\{\mathcal{F}_t\}$ is generated by a single Brownian motion. We note here that in terms of the degrees of freedom, the model we present here is equivalent to the first chaos theory. When the Wiener chaos coefficients for the random variable X_∞ take the special form

$$\phi_{s_1 s_2 \cdots s_n} = \phi_{s_1}\phi_{s_2}\cdots\phi_{s_n}, \tag{4.101}$$

for each n, where ϕ_s is a deterministic square–integrable function of one variable, we call the resulting interest rate system a *coherent term structure.* The term 'coherent' for this particular class of interest rate models has been first introduced in Brody and Hughston (2004) and accounts for similar mathematical techniques used in laser physics. In the case of a coherent term structure, we can make use of a formula due to Itô (1951) stating that

$$\int_0^s \int_0^{s_n} \cdots \int_0^{s_2} \phi_{s_1}\phi_{s_2}\cdots\phi_{s_n}\, dW_{s_1}dW_{s_2}\cdots dW_{s_n} = \|\phi\|_s^n H_n\left(\frac{\Phi_s}{\|\phi\|_s}\right), \tag{4.102}$$

where $\Phi_s = \int_0^s \phi_u dW_u$, $\|\phi\|_s^2 = \int_0^s \phi_u^2 du$, and $H_n(x)$ denotes the n–th Hermite polynomial

$$H_n(x) = \frac{1}{n!}(-1)^n e^{\frac{1}{2}x^2}\frac{d^n}{dx^n}\, e^{-\frac{1}{2}x^2}. \tag{4.103}$$

Using this identity and the orthogonality relation

$$\mathbb{E}\left[H_n(X)H_m(Y)\right] = \frac{1}{n!}\delta_{nm}\left(\mathbb{E}[XY]\right)^n, \tag{4.104}$$

satisfied by the Hermite polynomials for any pair of $N(0,1)$ Gaussian random variables X and Y, one can derive a closed form expression for the pricing kernel.

Another way to tackle the problem of explicitly representing the pricing kernel in this case, is to make the following observation. Let X_T be given by

$$X_T = \exp\left(\int_0^T f(s)dW_s\right) \tag{4.105}$$

for some fixed T, where $f(s)$ is a deterministic square–integrable function. We would like to determine the chaos expansion of this random variable. First we consider the identity

$$\exp\left(\int_0^T f(s)dW_s - \frac{1}{2}\|f\|_T^2\right) = \sum_{n=0}^{\infty} \|f\|_T^n H_n\left(\frac{F_T}{\|f\|_T}\right), \tag{4.106}$$

where $F_T = \int_0^T f(s)dW_s$ and $\|f\|_T^2 = \int_0^T f^2(s)ds$. As a consequence, the n–th coefficient in the chaos expansion of X_T in (4.105) is given by

$$\phi_{s_1 s_2 \cdots s_n} = \exp\left(\frac{1}{2}\|f\|_T^2\right) f(s_1)f(s_2)\cdots f(s_n). \tag{4.107}$$

As a result the generator of a coherent term structure is given by an expression of the form

$$X_\infty = \exp\left(\int_0^\infty \phi_s dW_s - \frac{1}{2}\int_0^\infty \phi_s^2 ds\right). \tag{4.108}$$

By use of Itô calculus we can then deduce that the conditional variance of X_∞ is

$$V_t = \left(\exp\left(\int_0^\infty \phi_s^2 ds\right) - \exp\left(\int_0^t \phi_s^2 ds\right)\right)\exp\left(2\int_0^t \phi_s dW_s - 2\int_0^t \phi_s^2 ds\right). \tag{4.109}$$

The first observation concerning the above expression is that the process V_t factorises into the product of two processes: an exponential martingale and a deterministic decreasing process. Thus, the associated bond price is deterministic, and is given by:

$$P_{tT} = \frac{1-\exp\left(-\int_T^\infty \phi_s^2 ds\right)}{1-\exp\left(-\int_t^\infty \phi_s^2 ds\right)}. \tag{4.110}$$

Likewise, the short rate is also deterministic, and is given by:

$$r_t = \frac{\phi_t^2}{\exp\left(\int_t^\infty \phi_s^2 ds\right) - 1}. \tag{4.111}$$

Concerning the problem of option pricing, we remark here that despite the fact that the interest rate system is deterministic, the valuation of an option still requires taking an expectation. This is because the pricing kernel process $\{V_t\}$

itself is stochastic. As an example we consider a European–style bond option with payoff

$$H_t = (P_{tT} - K)^+, \tag{4.112}$$

at time t. Then for $0 \leq s \leq t$ we have

$$\begin{aligned} C_s &= V_s^{-1}\mathbb{E}_s\left[V_t(P_{tT} - K)^+\right] \\ &= P_{st}(P_{tT} - K)^+, \end{aligned} \tag{4.113}$$

which is the correct expression for the payoff to be received at time t, discounted back to time s, where P_{tT} is given in (4.110).

We have observed so far that the family of chaos models termed coherent offer analytical tractability, although the resulting interest rate system is deterministic. We now show how one can use the results above in order to construct an essentially arbitrary stochastic term structure. This is based on the fact that a superposition of coherent models results to a stochastic term structure. To see this, we return to the general Wiener chaos expansion of square–integrable random variables. The space $\mathcal{H}$ spanned by square–integrable vectors of the form $(f, f_{s_1}, f_{s_1s_2}, f_{s_1s_2s_3}, \cdots)$, where the function $f_{s_1s_2s_3\cdots s_n}$ belongs to the space of a tensor product of n copies of a Hilbert space of square–integrable functions restricted to the subdiagonal domain for which $s_1 \geq s_2 \geq s_3 \geq \cdots \geq s_n$. The inner product space $\mathcal{H}$ is called a Fock space, and from the previous results we conclude that any admissible term structure can be represented as an element of $\mathcal{H}$. The coherent elements of $\mathcal{H}$, which can be expressed in the exponential form

$$C_\phi = (1,\, \phi_{s_1},\, \phi_{s_1}\phi_{s_2},\, \phi_{s_1}\phi_{s_2}\phi_{s_3},\, \cdots)\,, \tag{4.114}$$

play particularly important role in this regard, because a general element of $\mathcal{H}$ can be expressed as a combination of coherent vectors. That is to say, any interest rate model can be represented by a linear combination of a finite or infinite number of coherent term structure models.

An interesting result in this connection is that the inner product of a pair of coherent vectors in $\mathcal{H}$ is given by

$$\begin{aligned}(C_\phi, C_\psi) &= 1 + \int_0^\infty \phi_{s_1}\psi_{s_1} ds_1 + \int_0^\infty \int_0^{s_1} \phi_{s_1}\phi_{s_2}\psi_{s_1}\psi_{s_2} ds_1 ds_2 + \cdots \\ &= 1 + \int_0^\infty \phi_{s_1}\psi_{s_1} ds_1 + \frac{1}{2}\int_0^\infty \int_0^\infty \phi_{s_1}\phi_{s_2}\psi_{s_1}\psi_{s_2} ds_1 ds_2 + \cdots \\ &= \exp\left(\int_0^\infty \phi_s\psi_s ds\right), \end{aligned} \tag{4.115}$$

which shows that no pair of coherent vectors is orthogonal. Thus the space $\mathcal{C}$ of coherent vectors forms an overly complete basis for $\mathcal{H}$.

To demonstrate these results we give an example, namely the simplest nontrivial combination of coherent elements, given by $aC_\phi + bC_\psi$. A linear combination of coherent vectors cannot itself be expressed in the coherent form (4.114). Thus any realistic term structure is necessarily represented by an incoherent element of $\mathcal{H}$. In the case of two coherent term structures superimposed we derive a term structure that is manifestively stochastic. Moreover, the methodology can be applied for infinitely many coherent terms, as we shall demonstrate later. Because the space $\mathcal{H}$ is linear, when the chaos expansion is given by $aC_\phi + bC_\psi$, then the corresponding random variable takes the form

$$\begin{aligned}X_\infty &= a\exp\left(\int_0^\infty \phi_s dW_s - \frac{1}{2}\int_0^\infty \phi_s^2 ds\right) \\ &\quad + b\exp\left(\int_0^\infty \psi_s dW_s - \frac{1}{2}\int_0^\infty \psi_s^2 ds\right). \end{aligned} \tag{4.116}$$

For any square–integrable function f_s let us define an exponential martingale $\{M_t^f\}$ according to the scheme

$$M_t^f = \exp\left(\int_0^t f_s dW_s - \frac{1}{2}\int_0^t f_s^2 ds\right), \tag{4.117}$$

and a corresponding positive deterministic, decreasing function $\{\Delta_t^f\}$ by

$$\Delta_t^f = \exp\left(\int_0^\infty f_s^2 ds\right) - \exp\left(\int_0^t f_s^2 ds\right). \tag{4.118}$$

Then we can calculate the conditional variance of (4.116) to obtain:

$$V_t = a^2\Delta_t^\phi M_t^{2\phi} + b^2\Delta_t^\psi M_t^{2\psi} + 2ab\Delta_t^{\sqrt{\phi\psi}} M_t^{\phi+\psi}. \tag{4.119}$$

The corresponding expression for the bond price is

$$P_{tT} = \frac{a^2\Delta_T^{\phi}M_t^{2\phi} + b^2\Delta_T^{\psi}M_t^{2\psi} + 2ab\Delta_T^{\sqrt{\phi\psi}}M_t^{\phi+\psi}}{a^2\Delta_t^{\phi}M_t^{2\phi} + b^2\Delta_t^{\psi}M_t^{2\psi} + 2ab\Delta_t^{\sqrt{\phi\psi}}M_t^{\phi+\psi}}. \tag{4.120}$$

We note that the process $\{P_{tT}\}$ depends on the weighting factors a and b through the ratio a/b. As indicated above, stochasticity is inherent in an incoherent term structure.

Now we turn to the more general case of a chaos expansion given in the general form

$$X_\infty = \sum_k c_k C_{\phi_k}. \tag{4.121}$$

Then the pricing kernel takes the form

$$V_t = \sum_{k,l} c_k c_l \Delta_t^{\sqrt{\phi_k\phi_l}} M_t^{\phi_k+\phi_l}. \tag{4.122}$$

Making use of this formula we can then obtain explicit results for the discount bond system and other related processes of interest. The short rate is given by

$$r_t = \frac{\sum_{k,l} c_k c_l \phi_{kt}\phi_{lt} \exp\left(\int_0^t \phi_{ks}\phi_{ls}ds\right) M_t^{\phi_k+\phi_l}}{\sum_{k,l} c_k c_l \Delta_t^{\sqrt{\phi_k\phi_l}} M_t^{\phi_k+\phi_l}}. \tag{4.123}$$

The market price of risk vector can be expressed in the form

$$\lambda_t = -\frac{\sum_{k,l} c_k c_l (\phi_{kt}+\phi_{lt}) \Delta_t^{\sqrt{\phi_k\phi_l}} M_t^{\phi_k+\phi_l}}{\sum_{k,l} c_k c_l \Delta_t^{\sqrt{\phi_k\phi_l}} M_t^{\phi_k+\phi_l}}. \tag{4.124}$$

The volatility processes for the discount bonds is given by

$$\Omega_{tT} = \lambda_t + \frac{\sum_{k,l} c_k c_l (\phi_{kt}+\phi_{lt}) \Delta_T^{\sqrt{\phi_k\phi_l}} M_t^{\phi_k+\phi_l}}{\sum_{k,l} c_k c_l \Delta_T^{\sqrt{\phi_k\phi_l}} M_t^{\phi_k+\phi_l}}. \tag{4.125}$$

By use of expression (4.124) we see that the maturity condition $\Omega_{TT} = 0$ is satisfied. The initial term structure P_{0T} is given by

$$P_{0T} = \frac{\sum_{k,l} c_k c_l \Delta_T^{\sqrt{\phi_k\phi_l}}}{\sum_{k,l} c_k c_l \left(\exp(\int_0^\infty \phi_{ks}\phi_{ls}ds) - 1\right)}, \tag{4.126}$$

and the initial short rate r_0 is

$$r_0 = \frac{\sum_{k,l} c_k c_l \phi_{k0}\phi_{l0}}{\sum_{k,l} c_k c_l \left(\exp(\int_0^\infty \phi_{ks}\phi_{ls}ds) - 1\right)}. \tag{4.127}$$

The pricing of derivatives is also possible. For example, the value at time s of a bond option with payoff

$$H_t = (P_{tT} - K)^+, \tag{4.128}$$

is given by the following

$$C_s = V_s^{-1} E_s \left[\left(\sum_{k,l} c_k c_l \left(\Delta_T^{\sqrt{\phi_k \phi_l}} - K \Delta_t^{\sqrt{\phi_k \phi_l}} \right) M_t^{\phi_k + \phi_l} \right)^+ \right]. \tag{4.129}$$

This expression is similar to the valuation formula for a standard basket option, for which various well–established numerical methods have been developed (see, e.g., Turnbull & Wakeman 1991, Curran 1994).

Although linear combinations of coherent elements of $\mathcal{H}$ is sufficient to cover essentially all admissible term structures, one is not confined to adopt the coherent state decomposition exclusively for generating interest rate models. For instance, coherent elements can be combined with finite chaos elements. We indicate here an example, discussed by Brody and Hughston (2004).

Consider a chaos expansion of X_∞ in terms of the combination of a coherent element C_ϕ and a first chaos vector of $\mathcal{H}$ of the form $(0, \alpha_s, 0, 0, \cdots)$ in the single–factor model. It follows from the linearity of $\mathcal{H}$ that

$$X_\infty = \int_0^\infty \alpha_s dW_s + \exp\left(\int_0^\infty \phi_s dW_s - \frac{1}{2} \int_0^\infty \phi_s^2 ds \right). \tag{4.130}$$

We now have to calculate the pricing kernel process, that is, we need to calculate the conditional variance of the random variable X_∞. The contributions arising from each term in the right–hand side of (4.130) can be determined in a straightforward way. However, to calculate the cross term we introduce an auxiliary variable χ and write

$$\begin{aligned} &E_t \left[(\textstyle\int_0^\infty \alpha_s dW_s) \exp\left(\displaystyle\int_0^\infty \phi_s dW_s - \frac{1}{2} \int_0^\infty \phi_s^2 ds \right) \right] \\ &= \frac{\partial}{\partial \chi} E_t \left[\exp\left(\int_0^\infty (\phi_s + \chi \alpha_s) dW_s - \frac{1}{2} \int_0^\infty \phi_s^2 ds \right) \right] \Bigg|_{\chi=0}. \end{aligned} \tag{4.131}$$

Following the calculation above, we find that the bond price in this model is given by

$$P_{tT} = \frac{\int_T^\infty \alpha_s^2 ds + \Delta_T^\phi M_t^{2\phi} + (\int_T^\infty \alpha_s \phi_s ds) M_t^\phi}{\int_t^\infty \alpha_s^2 ds + \Delta_t^\phi M_t^{2\phi} + (\int_t^\infty \alpha_s \phi_s ds) M_t^\phi}, \tag{4.132}$$

where we use the definitions (4.117) and (4.118). In the special case for which the supports of the functions α_s and ϕ_s share no overlap, then we recover a rational lognormal model. The method used in (4.131) is effective in general when we consider models that are generated by combinations of finite and coherent elements.

4.12 Relation to information geometry

We remark, incidentally, that there is a link between the chaos structure presented here and the applications of information geometry in interest rate theory proposed in Brody and Hughston (2001a,b, 2002). It is shown there that the system of positive interest term structures is isomorphic to the space of density functions on the positive real line. This is a natural consequence of the observation that the initial term structure can be viewed as the tail of a probability distribution, once we impose the positivity conditions

$$0 < P_{0T} \leq 1, \tag{4.133}$$

and

$$\frac{\partial P_{0T}}{\partial T} < 0. \tag{4.134}$$

Then the function

$$\rho(T) = -\frac{\partial P_{0T}}{\partial T}, \tag{4.135}$$

is the corresponding probability density function, which we call the term structure density, the square root of which can be regarded as a unit vector element of the

Hilbert space $H = L^2(R^1_+)$. By using results from information geometry, one can measure how different two term structures are. More precisely, given two yield curves, we can introduce a measure of the distance between them. When the representation of the yield curve is in a non–parametric form, this can be carried out, for example, by use of the Bhatacharyya spherical distance

$$\theta(\rho_1, \rho_2) = \cos^{-1} \int_0^\infty \sqrt{\rho_1(T)\rho_2(T)}dT, \tag{4.136}$$

where ρ_1 and ρ_2 are the term structure densities arising from the two yield curves. In the parametric case, one can use the Fisher–Rao metric associated with the given parametric family. In a first chaos model we can without loss of generality normalise the integral of the square of ϕ_s to unity. Then it follows immediately from (4.5) that the square of ϕ_s is the associated term structure density.

Chapter 5

General asset price dynamics and option pricing

5.1 Foreign exchange systems

Now we proceed to consider how the framework presented so far generalises to the situation where there is a foreign exchange system, with a family of discount bonds associated to each currency. It will be demonstrated that a chaotic representation exists for the entirety of such an international system of interest rates and foreign exchange. As a byproduct of this result, we are also led to a simple class of stochastic volatility models for general asset price dynamics. Motivated by this fact we provide some examples of specific models in this framework for asset price dynamics and option pricing. The core of the analysis in this chapter, is (a) the generic representation of a foreign exchange process as a ratio of conditional variances; and (b) the realisation that this result holds also for a one currency economy, as long as we interpret the associated rates as dividend streams, and we restrict ourselves to assets of limited liability. The representation of the foreign exchange system as a ratio of numeraires has been discussed by a number of authors (see e.g., Flesaker and Hughston 1997, Lipton 2001, Rogers 1997, Saa-Requejo 1993). Here we show how this can be deduced from our axiomatic framework and how it translates to a representation in terms of the conditional variances of two square–integrable random variables on Wiener space. This is based on the conditional variance representation of the pricing kernel process. We will show how this generalises to a multi–currency situation. That is to say, a conditional variance representation exists for the pricing kernel associated with each currency. Then we proceed to provide various examples, interpreting the results in a one–currency setting.

The problem of characterising the general dynamics of an arbitrage–free system of interest rates and foreign exchange was first considered by Amin and Jarrow (1992) in the context of generalising the HJM framework. For convenience we shall in the discussion that follows adopt the conventions of Flesaker and Hughston (1997), and write $\{S_t^{ij}\}$ for the price process of one unit of currency i in units of currency j. Here $i, j = 0, 1, \cdots, N$, and we may think of the case $i = 0$ as referring to the base currency with respect to which the axioms (A1), (A2), (A3) and (A4) are framed. There is no special significance to the choice of base currency: the entire system is symmetrical in the ensemble of currencies. In the present investigation the foreign exchange market is taken to be frictionless in the sense that $\{S_t^{ij} S_t^{jk}\} = \{S_t^{ik}\}$ for all i, j, k.

Let us write $\{B_t^i\}$ for the value in units of currency i of the money–market account process in that currency, initialised to one unit of currency i. It will be assumed that for each currency there exists a strictly increasing money–market asset, with a corresponding strictly positive short rate process $\{r_t^i\}$ such that

$$B_t^i = B_0^i \exp\left(\int_0^t r_s^i ds\right). \tag{5.1}$$

We also assume the existence of a floating rate note in each currency: that is to say, for each i it will be assumed that there exists an asset of constant value in units of currency i, paying a dividend at the rate with process $\{r_t^i\}$.

Writing $\{S_t^{i0}\}$ for the value of one unit of currency i in units of the base currency, we see that the product $\{S_t^{i0} B_t^i\}$ represents the base–currency price of a non–dividend paying asset. Therefore by axiom (A2) we deduce for each i that $\{S_t^{i0}B_t^i/\xi_t\}$ is a martingale, and it follows that $\{S_t^{i0}/\xi_t\}$ is a supermartingale (since $\{B_t^i\}$ is an increasing process). Thus if we define

$$V_t^i = \frac{S_t^{i0}}{\xi_t}, \tag{5.2}$$

then since

$$S_t^{ij} S_t^{j0} = S_t^{i0}, \tag{5.3}$$

for all i, j, we deduce that

$$S_t^{ij} = \frac{V_t^i}{V_t^j}. \tag{5.4}$$

This gives us a general expression for the exchange–rate process as a ratio of supermartingales (Rogers 1997, Flesaker and Hughston 1997). As a consequence we deduce that the dynamics of $\{S_t^{ij}\}$ are given by

$$\frac{dS_t^{ij}}{S_t^{ij}} = \left[r_t^j - r_t^i + \lambda_t^j\left(\lambda_t^j - \lambda_t^i\right)\right] dt + \left(\lambda_t^j - \lambda_t^i\right) dW_t, \tag{5.5}$$

where $\{\lambda_t^i\}$ is the market price of risk vector process associated with assets that are denominated in currency i. The derivation of (5.5) follows directly from (5.4) and the relation

$$dV_t^i = -r_t^i V_t^i dt - \lambda_t^i V_t^i dW_t, \tag{5.6}$$

together with the Itô quotient rule (4.20). It is interesting to note that in the general arbitrage–free exchange rate dynamics the volatility is completely determined by the associated market price of risk processes (Flesaker and Hughston 1997, for further discussion see also Lipton 2001).

Let us now consider the discount bond system for foreign currency number i. We denote by P_{tT}^i the value at time t of a bond that pays one unit of currency i at time T. In this case $\{S_t^{i0}P_{tT}^i\}$ is the base–currency price of a non–dividend paying asset, and therefore the process $\{S_t^{i0} P_{tT}^i / \xi_t\}$ is a martingale by (A2). It follows that

$$S_t^{i0} P_{tT}^i / \xi_t = E_t \left[S_T^{i0} P_{TT}^i / \xi_T \right]. \tag{5.7}$$

Thus, since

$$\frac{S_t^{i0}}{\xi_t} = V_t^i, \tag{5.8}$$

and $P_{TT}^i = 1$, we deduce that

$$P_{tT}^i = \frac{E_t \left[V_T^i \right]}{V_t^i}. \tag{5.9}$$

Now we make the assumption that

$$\lim_{T \to \infty} P_{0T}^i = 0, \tag{5.10}$$

for all i. We infer then that a conditional variance representation exists for the pricing kernel associated with each currency. In other words, there exists a set of random variables $X_\infty^i \in L^2(\Omega, \mathcal{F}, P)$ for $i = 0, 1, \cdots, N$ such that

$$V_t^i = E_t \left[\left(X_\infty^i - E_t \left[X_\infty^i \right] \right)^2 \right]. \tag{5.11}$$

These random variables then each admit a chaos representation in terms of the vector Wiener process $\{W_t^\alpha\}$ ($\alpha = 1, \cdots, k$). We see that once the random variables X_∞^i have been specified for $i = 0, 1, \cdots, N$ then the international system of interest and foreign exchange is completely determined by (5.4), (5.9) and (5.11). We can refer to the random variables X_∞^i as the 'generators' of the corresponding interest rate and foreign exchange system.

It should be evident that although we have consistently used the language of foreign exchange in the discussion above, the matrix process $\{S_t^{ij}\}$ can be used to characterise the price of any asset in terms of another, providing that these prices are always positive and that we interpret the associated short–rate systems as continuous dividend streams. As a consequence we see that the 'generic' model for an asset price is a process of the form

$$S_t = \frac{E_t\left[(Y_\infty - E_t\left[Y_\infty\right])^2\right]}{E_t\left[(X_\infty - E_t\left[X_\infty\right])^2\right]}, \tag{5.12}$$

that is to say, a ratio of conditional variances, where X_∞ and Y_∞ are elements of $L^2(\Omega, \mathcal{F}, P)$. Equation (5.12) is in essence a different form of (5.4), rewritten in the language of a one–currency economy, and using the conditional variance representation for the pricing kernel process. We explore this property in the next section.

5.2 General assets and option pricing

The chaotic representation is still valid in the case of assets of limited liability. This observation is the starting point of the present analysis, and we begin by rewriting the generic formula (5.12) for the price trajectories of an asset in terms of the quotient of two conditional variances:

$$S_t = \frac{\mathrm{Var}_t[Y_\infty]}{\mathrm{Var}_t[X_\infty]}. \tag{5.13}$$

It is evident that we need two different chaos expansions, one for each of the generators Y_∞ and X_∞. The connection with the foreign exchange systems is immediate: one considers the associate short rates as continuous dividend streams. Now, the general dynamics for a limited liability asset price process $\{S_t\}$ paying a dividend rate $\{\delta_t\}$ are

$$dS_t = (r_t - \delta_t + \lambda_t\sigma_t)S_t dt + \sigma_t S_t dW_t. \tag{5.14}$$

A comparison with (5.5) gives us immediately the relation between the parameters in a foreign exchange and in a general assets context; we interpret the short rate and the market price of risk of economy j as the ones of the domestic economy, and the dividend rate is represented by the rate of the 'foreign' economy i. The volatility of the asset with price process $\{S_t\}$ is given by the difference of the market prices of risk, i.e. $\sigma_t = \lambda_t^j - \lambda_t^i$. In connection to derivatives pricing we will always need to find explicitly the pricing kernel process $\{V_t\}$.

5.3 Combination of first chaos models

The case of a combination of first chaos expansions for both generators X_∞ and Y_∞ is a simple one, since the resulting model is one of a deterministically evolving asset. Nevertheless the model is important since it represents the general deterministic notion of a continuous price processes compatible with the principles of no arbitrage. Consider the case of two generators that have a first chaos expansion each, with the square–integrable functions ψ_s and ϕ_s for Y_∞ and X_∞ respectively. The asset price $\{S_t\}$ follows a deterministic process given by

$$S_t = \frac{\int_t^\infty \psi_s^2 ds}{\int_t^\infty \phi_s^2 ds}. \tag{5.15}$$

The function ϕ_s can be calibrated from the short rate since we have here again

$$r_t = \frac{\phi_t^2}{\int_t^\infty \phi_s^2 ds}. \tag{5.16}$$

The dividend rate process is determined by Y_∞ and is given by

$$\delta_t = \frac{\psi_t^2}{\int_t^\infty \psi_s^2 ds}. \tag{5.17}$$

The initial value of the asset price is

$$S_0 = \frac{\int_0^\infty \psi_s^2 ds}{\int_0^\infty \phi_s^2 ds}. \tag{5.18}$$

The price of a European call is

$$H_0 = \exp\left(-\int_0^T r_s ds\right)\left(\frac{\int_T^\infty \psi_s^2 ds}{\int_T^\infty \phi_s^2 ds} - K\right) \quad \text{if} \quad S_T \geq K\,, \tag{5.19}$$

and zero otherwise.

5.4 Combination of first chaos and coherent system

We consider here the case of a coherent chaos expansion for the generator Y_∞, characterised by the deterministic function ϕ_t and a first chaos expansion for the generator X_∞, characterised by the deterministic function ψ_t. This example has been studied also in Brody and Hughston (2004). Here we take the analysis further to include some explicit formulae for option prices. We represent the random variable X_∞ in the form

$$X_\infty = \int_0^\infty \psi_s dW_s, \tag{5.20}$$

whereas the random variable Y_∞ is set to take the form (cf. section 4.11)

$$Y_\infty = \exp\left(\int_0^\infty \phi_s dW_s - \frac{1}{2}\int_0^\infty \phi_s^2 ds\right). \tag{5.21}$$

We now make use of expression (5.13) and the result is the following representation for a limited liability, dividend paying asset:

$$S_t = S_0 \exp\left(\int_0^t \sigma_s dW_s - \frac{1}{2}\int_0^t \sigma_s^2 ds + \int_0^t (r_s - \delta_s) ds\right), \tag{5.22}$$

where we have set $\phi_t = \frac{1}{2}\sigma_t$ for a square–integrable deterministic function σ_t which we identify as the volatility. In connection with the remarks in the previous section we see that

$$\sigma_t = \lambda_t^j - \lambda_t^i, \tag{5.23}$$

where $\lambda_t^j = 0$ (the market price of risk associated to the first chaos) and $\lambda_t^i = -2\phi_t$ (the market price of risk associated to the coherent chaos). The initial value of the asset price process $\{S_t\}$ is

$$S_0 = \frac{\exp\left(\frac{1}{4}\int_0^\infty \sigma_s^2 ds\right) - 1}{\int_0^\infty \psi_s^2 ds}. \tag{5.24}$$

For the short rate we have

$$r_t = \frac{\psi_t^2}{\int_t^\infty \psi_s^2 ds}, \tag{5.25}$$

and for the dividend rate we have

$$\delta_t = \frac{\frac{1}{4}\sigma_t^2}{\exp\left(\frac{1}{4}\int_t^\infty \sigma_s^2 ds\right) - 1}. \tag{5.26}$$

There are two ways to establish expression (5.22), along with the representations for the short rate process, the volatility process, and the market price of risk. One could consider the explicit form of the pricing kernel in the case of a coherent expansion which we reproduce here

$$\mathrm{Var}_t[Y_\infty] = \left(\exp\left(\int_0^\infty \phi_s^2 ds\right) - \exp\left(\int_0^t \phi_s^2 ds\right)\right) \exp\left(2\int_0^t \phi_s dW_s - 2\int_0^t \phi_s^2 ds\right), \tag{5.27}$$

in combination with the conditional variance representation of a first chaos expansion with parameter ψ_s, which, as shown in section 4.1, takes the simple form

$$\mathrm{Var}_t[X_\infty] = \int_t^\infty \psi_s^2 ds. \tag{5.28}$$

We therefore have:

$$S_t = \frac{\left(\exp\left(\int_0^\infty \phi_s^2 ds\right) - \exp\left(\int_0^t \phi_s^2 ds\right)\right) \exp\left(2\int_0^t \phi_s dW_s - 2\int_0^t \phi_s^2 ds\right)}{\int_t^\infty \psi_s^2 ds}. \tag{5.29}$$

It is then a matter of rearranging the above formula in order to derive (5.22), and then by use of Itô's formula to produce the standard no–arbitrage dynamics for the asset price.

Alternatively, one can solve equation (5.5) by use of Itô's rule, and take into consideration the fact that we identify the short rate process $\{r_t\}$ with the short rate of economy j, and the dividend rate process with the short rate of economy i. The volatility of the asset is then $\{\sigma_t\} = \{\lambda_t^j - \lambda_t^i\}$. The point is that we already know these expressions from the considerations of interest rate modelling in previous chapters, and we do not need to explicitly perform the calculations.

To be able to price options one needs to specify what is the pricing kernel associated with this model. This is not difficult once we notice that the market price of risk is zero and combine this with the formula above for the short rate process $\{r_t\}$. Then the pricing kernel is the discount factor connected to the first chaos:

$$V_t = V_0 \exp\left(-\int_0^t r_s ds\right), \qquad \text{with} \qquad V_0 = \int_0^\infty \psi_s^2 ds. \tag{5.30}$$

We see here that the pricing kernel is deterministic, as one would expect from a first chaos model. In fact, it is not difficult to show that expression (5.30) is equal to the familiar representation for the conditional variance of a first chaos expanded random variable (see equation (4.2)). We have:

$$\begin{aligned}
V_t &= \int_0^\infty \psi_s^2 ds \exp\left(-\int_0^t \frac{\psi_s^2}{\int_s^\infty \psi_u^2 du} ds\right) \\
&= \int_0^\infty \psi_s^2 ds \exp\left(\int_0^t \left(\ln\left(\int_s^\infty \psi_u^2 du\right)\right)' ds\right) \\
&= \int_0^\infty \psi_s^2 ds \exp\left(\ln\left(\frac{\int_t^\infty \psi_u^2 du}{\int_0^\infty \psi_u^2 du}\right)\right) \\
&= \int_t^\infty \psi_s^2 ds.
\end{aligned} \tag{5.31}$$

The pricing of European derivatives is now straightforward, for example the price of a vanilla call with payoff function $H_T = (S_T - K)^+$ at expiry T for some strike K is

$$H_0 = \exp\left(-\int_0^T r_s ds\right) E\left[\left(U_T^{(1)} \exp\left(\int_0^T \sigma_s dW_s\right) - K\right)^+\right], \tag{5.32}$$

where we denote by $U_t^{(1)}$ the deterministic function

$$U_t^{(1)} = S_0 \exp\left(-\frac{1}{2}\int_0^t \sigma_s^2 ds + \int_0^t (r_s - \delta_s) ds\right). \tag{5.33}$$

The argument for calculating the expectation above is essentially the same as the one in the usual Black–Scholes analysis. We simply make use of the fact that the only state variable here is a normally distributed random variable. The value we get for the European call is then the standard Black–Scholes expression:

$$H_0 = \exp\left(-\int_0^T \delta_s ds\right) S_0 N(d_1) - K \exp\left(-\int_0^T r_s ds\right) N(d_2), \tag{5.34}$$

where we define the volatility parameter

$$M_t = \int_0^t \sigma_s^2 ds, \tag{5.35}$$

and the usual Black–Scholes parameters:

$$\begin{aligned} d_1 &= \frac{\log\left(\frac{S_0}{K}\right) + \frac{1}{2}M_T + \int_0^T (r_s - \delta_s)\, ds}{\sqrt{M_T}}, \\ d_2 &= d_1 - \sqrt{M_T}. \end{aligned} \tag{5.36}$$

and N is the cumulative distribution function of a standard normally distributed random variable. It is now possible to derive expressions for the sensitivity parameters; for instance, we can calculate the delta to be

$$\frac{\partial H_0}{\partial S_0} = \exp\left(-\int_0^T \delta_s ds\right) N(d_1). \tag{5.37}$$

Any derivative of European type can be analytically priced under this framework. This specific example of the combination of a first chaos and coherent expansions is one of the simplest we can use; however we could consider it to be equivalent to Black–Scholes, as far as the degrees of freedom are concerned. That is to say, under the above considerations, the economy supports a stochastic asset with a price process $\{S_t\}$ and a deterministic risk free money market account. Moreover, the formulae for standard European options are identical to Black–Scholes, as shown above.

5.5 Combination of coherent systems

Another model for an asset price emerges when we consider the case of two coherent systems, one for each generator. Let us therefore consider the case where both Y_∞ and X_∞ are characterised by a coherent chaos expansion with functions ϕ_t and ψ_t respectively. This is a more general model than the previous one since there is a non–deterministic pricing kernel. We shall consider the case of a k–dimensional Wiener process, and therefore ϕ_t and ψ_t should be thought as vectors. By consideration of (5.13), we deduce that the asset price process $\{S_t\}$ can be written in the form

$$S_t = S_0 \exp\left(\int_0^t \sigma_s dW_s - \frac{1}{2}\int_0^t \sigma_s^2 ds + \int_0^t (r_s - \delta_s)ds + \int_0^t \lambda_s \sigma_s ds\right), \quad (5.38)$$

with

$$S_0 = \frac{\exp\left(\int_0^\infty \phi_s^2 ds\right) - 1}{\exp\left(\int_0^\infty \psi_s^2 ds\right) - 1}. \quad (5.39)$$

As mentioned earlier, the fact that we do not need to explicitly perform this calculation is because we know already what the pricing kernel is in the case of a coherent chaos expansion (cf. section 4.9). Indeed this is the case not only for here, but for all the other examples that follow. The same holds for the short rate, the dividend rate and the market price of risk processes. In what follows in this section we use the same notation as before for the short rates, volatility, dividend stream, and market price of risk. In particular, we have

$$r_t = \frac{\psi_t^2}{\exp\left(\int_t^\infty \psi_s^2 ds\right) - 1}, \qquad \delta_t = \frac{\phi_t^2}{\exp\left(\int_t^\infty \phi_s^2 ds\right) - 1}, \quad (5.40)$$

for the short rate and the dividend, which are the same as before, and also

$$\lambda_t = -2\psi_t, \qquad \sigma_t = 2(\phi_t - \psi_t), \quad (5.41)$$

for the market price of risk and the volatility respectively. Consequently, the pricing kernel is of the form

$$V_t = V_0 \exp\left(-\int_0^t (r_s + 2\psi_s^2)ds + 2\int_0^t \psi_s dW_s\right), \quad (5.42)$$

where

$$V_0 = \exp\left(\int_0^\infty \psi_s^2 ds\right) - 1. \quad (5.43)$$

It is an exercise to combine expressions (5.42) and (5.43) along with the representation of the short rate in (5.40) in order to derive the familiar expression for the pricing kernel in the case of a coherent chaos, given in (4.109).

To continue, we note that we do not consider the initial values of the pricing kernels to be one. In principle we could do this for one pricing kernel only. Had we impose this condition to both conditional variances we would have $S_0 = 1$, i.e. unit initial value for the asset S_t, and there is no reason why we should restrict to this case. The same argument holds when we model the foreign exchange dynamics between two economies under the same framework. In general, when we consider an asset, it is an unrealistic assumption to start it at value one. We were able to consider the initial value of the pricing kernel to be one in the term structure case, in previous chapters, because of the explicit form of dependence of the discount bond system to this process. In other words, and by consideration of expression (3.39) we see that setting $V_0 = 1$ does not imply that the initial value of the bond is one (except when the maturity of the bond is also zero). It is because of the very nature of the discount bond system as a 'specific' asset whose value by construction depends on two different points in the future, that we are able to simplify the analysis in the interest rate modelling case by setting $V_0 = 1$.

Returning to the discussion about option pricing in the current model, it will be helpful to comment that expression (5.42) is nothing but the pricing kernel given by a coherent expansion of the vector deterministic process ψ_t. This can be shown by a substitution of the rate r_t after which we are able to derive the familiar expression (4.109).

The pricing of a European call option in this model is somewhat different, though still a mater of performing Gaussian calculations. We base our analysis to the standard pricing formula

$$H_0 = E\left[\frac{V_T}{V_0}(S_T - K)^+\right]. \tag{5.44}$$

By consideration of (5.42) and (5.38), this translates into the expression

$$H_0 = M_T E\left[\exp\left(2\int_0^T \psi_s dW_s\right)\left(S_0 L_T \exp\left(\int_0^T \sigma_s dW_s\right) - K\right)^+\right], \tag{5.45}$$

where for convenience we make use of the following deterministic processes:

$$
\begin{aligned}
M_t &= \exp\left(-\int_0^t \left(r_s + 2\psi_s^2\right) ds\right) \\
L_t &= \exp\left(\int_0^t \left(r_s - \delta_s\right) ds + \int_0^t \lambda_s \sigma_s ds - \frac{1}{2}\int_0^t \sigma_s^2 ds\right).
\end{aligned} \tag{5.46}
$$

Recall here that indeed, in this model, the volatility along with the short rate process and the risk premium are all deterministic; therefore the only stochastic terms in the expectation (5.45) are the following normally distributed Gaussian martingales:

$$
R_t = 2\int_0^t \phi_s dW_s, \qquad \bar{R}_t = 2\int_0^t \psi_s dW_s, \tag{5.47}
$$

for which the associated quadratic variations are:

$$
Q_t = 4\int_0^t \phi_s^2 ds, \qquad \bar{Q}_t = 4\int_0^t \psi_s^2 ds, \tag{5.48}
$$

respectively. Now we make use of the fact that $\sigma_s = 2(\phi_s - \psi_s)$, and the evaluation of the initial price for the derivative, in view of (5.45), translates to the calculation of the following expectation

$$
H_0 = M_T E\left[\left(S_0 L_T e^{X_T\sqrt{Q_T}} - K e^{Y_T\sqrt{\bar{Q}_T}}\right)^+\right], \tag{5.49}
$$

where X and Y are standard normally distributed random variables, given by the following expressions:

$$
X_T = \frac{R_T}{\sqrt{Q_T}}, \qquad Y_T = \frac{\bar{R}_T}{\sqrt{\bar{Q}_T}}. \tag{5.50}
$$

These random variables are dependent, with correlation coefficient $\rho = E[XY]$. We therefore consider the following integral:

$$
H_0 = M_T \int_{-\infty}^{\infty}\int_{y^*}^{\infty} \left(S_0 L_T e^{x\sqrt{Q_T}} - K e^{y\sqrt{\bar{Q}_T}}\right) f(x,y)\,dx\,dy. \tag{5.51}
$$

Here $f(x,y)$ is the standard bivariate normal density function

$$
f(x,y) = \frac{1}{2\pi\sqrt{1-\rho}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[x^2 + y^2 - 2\rho xy\right]\right), \tag{5.52}
$$

and y^* is defined according to the following formula:

$$y^* = \frac{\log\left(\frac{K}{S_0 L_T}\right) + y\sqrt{\bar{Q}_T}}{\sqrt{Q_T}}, \tag{5.53}$$

which is equivalent to:

$$y^* = \frac{\log(K/S_0) - \int_0^T (r_s - \delta_s)ds - \int_0^T \lambda_s \sigma_s ds + \frac{1}{2}\int_0^T \sigma_s^2 ds + y\sqrt{\bar{Q}_T}}{\sqrt{Q_T}}. \tag{5.54}$$

Similar semi–analytic expressions can be found for exotic European style options where the payoff depends on an arbitrary function of S_T.

5.6 Combination of first and second chaos

Now we consider the model that arises in the case of a second chaos expansion for the generator Y_∞ and a first chaos expansion for X_∞ in equation (5.13). In particular, we tackle the situation where the random variable Y_∞ admits a second factorisable chaos expansion with deterministic coefficients α_s, β_s, and γ_s, and the denominator is characterised by a first chaos with function ψ_s. The second factorisable chaos model is studied in detail in section 4.5 in the context of interest rate modelling. Here we are using the same notation and additionally we define the functions

$$\widetilde{A}_t = \int_0^t (\alpha_s^2 + \beta_s^2 Q_s) ds, \tag{5.55}$$

and

$$D_t = \int_t^\infty \psi_s^2 ds. \tag{5.56}$$

It will be convenient here to reintroduce the following deterministic functions:

$$\begin{aligned} A_t &= \int_t^\infty \left(\alpha_s^2 + \beta_s^2 Q_s\right) ds, \\ B_t &= 2\int_t^\infty \alpha_s \beta_s ds, \\ C_t &= \int_t^\infty \beta_s^2 ds. \end{aligned} \tag{5.57}$$

which will come into play again shortly. As in the first example, in this model, the stochasticity involved will be coming from only the chaos expansion of Y_∞, since the first chaos expansion for the random variable X_∞ gives rise to a deterministic pricing kernel. In the analysis to follow we are using the Gaussian martingale $\{R_t\}$ along with its quadratic variation $\{Q_t\}$

$$R_t = \int_0^t \gamma_s dW_s, \qquad Q_t = \int_0^t \gamma_s^2 ds. \tag{5.58}$$

The model for the stochastic evolution of the asset price process $\{S_t\}$ can then be put in the form

$$S_t = \frac{1}{D_t}\left(S_0 D_0 - \widetilde{A}_t + B_t R_t + C_t\left(R_t^2 - Q_t\right)\right). \tag{5.59}$$

In this case we find that $\{S_t\}$ satisfies a stochastic differential equation of the form:

$$\frac{dS_t}{S_t} = (r_t - \delta_t)\, dt + \sigma_t dW_t. \tag{5.60}$$

Equation (5.60) is essentially the same as the general stochastic dynamics described in (5.5) for the evolution of the foreign exchange between two economies. This observation lies in the core of the approach we consider here and we see that the interest rate dynamics is the rate of the 'domestic' economy, whereas what we consider here as the dividend rate process is the short rate for the 'foreign' economy. What we call the domestic economy in the foreign exchange language, comes into play under the current considerations with the expansion of the random variable X_∞. This fact explains why there is no 'volatility' term in the drift, in equation (5.60). Since the market price of risk vector in the first chaos framework is zero the drift term becomes as in (5.60). On the other hand to define the volatility for the asset price process $\{S_t\}$, which coincides with the volatility of the foreign exchange in the multiple economy language, one is lead to consider the risk premium vector for the expansion of Y_∞ as well, which is known by consideration of (4.22).

Based on the above discussion, we conclude that the short rate process is deterministic and given by the representation

$$r_t = \frac{\psi_t^2}{D_t}, \tag{5.61}$$

whereas the dividend and volatility for the asset price $\{S_t\}$ are

$$\delta_t = \frac{(\alpha_t + \beta_t R_t)^2}{A_t + B_t R_t + C_t(R_t^2 - Q_t)}, \tag{5.62}$$

and

$$\sigma_t = \frac{\gamma_t B_t + 2\gamma_t C_t R_t}{A_t + B_t R_t + C_t(R_t^2 - Q_t)}. \tag{5.63}$$

We now consider the problem of option pricing. Again here we will consider a European call option with payoff $(S_T - K)^+$ for some strike K; other options can also be considered. The pricing of European options is important, since these are typically the liquid assets in the market, and therefore serve as basic calibration tool for the pricing of exotic derivatives. To be able to derive the current price of a European Call we need to consider the expectation

$$H_0 = E\left[\frac{V_T}{V_0}(S_T - K)^+\right]. \tag{5.64}$$

Again here the pricing kernel is the one arising in the first chaos context. Based on the previous expressions for the asset price process $\{S_t\}$ and noting that $S_0 D_0 - \widetilde{A}_t = A_t$, we conclude that the expression in the expectation is of the form:

$$H_0 = \frac{e^{-\int_0^T r_s ds}}{D_T} E\left[\left(A + BZ_T + CZ_T^2\right)^+\right], \tag{5.65}$$

where we now define, for this section only,

$$\begin{aligned} A &= A_T - C_T Q_T - K D_T \\ B &= B_T\sqrt{Q_T}, \qquad C = C_T Q_T, \qquad Q_T = \int_0^T \gamma_s^2 ds, \end{aligned} \tag{5.66}$$

and $Z_T = R_T/\sqrt{Q_T}$ is a standard normally distributed random variable. The deterministic expression outside the expectation in (5.65) is $1/V_0$, where V_0 is the initial value for the pricing kernel implied by the first chaos expansion of X_∞, that is

$$\frac{e^{-\int_0^T r_s ds}}{D_T} = \frac{1}{V_0} = \frac{1}{\int_0^\infty \psi_s^2 ds}, \tag{5.67}$$

where $\{D_t\}$ is defined in (5.56). The exponential in the above expression is what usually appears in the derivatives pricing literature as the 'discount factor', when one considers the money market account as the numeraire. We choose here to keep this notation and not substitute for the simple expression on the right hand side of (5.67). The option pricing formula now becomes

$$H_0 = \frac{e^{-\int_0^T r_s ds}}{D_T\sqrt{2\pi}} \int_{\mathcal{P}_1(z)\geq 0} \mathcal{P}_1(z)dz. \tag{5.68}$$

As in section 4.5 we consider all different cases for the polynomial

$$\mathcal{P}_1(z) = A + Bz + Cz^2. \tag{5.69}$$

This leads us to the consideration of the sign of the discriminant of this polynomial, given by

$$\Pi = B_T^2 Q_T + 4C_T Q_T \left(KD_T + C_T Q_T - A_T\right). \tag{5.70}$$

When $\Pi \leq 0$ then the option is guaranteed to expire in the money, and the current price is

$$H_0 = \exp\left(-\int_0^T r_s ds\right)\left(\frac{S_0 D_0 - \widetilde{A}_T}{D_T} - K\right). \tag{5.71}$$

Then the delta of this position is

$$\frac{\partial H_0}{\partial S_0} = \frac{\exp\left(-\int_0^T r_s ds\right)}{\int_T^\infty \psi_s^2 ds} = \frac{1}{\int_0^\infty \psi_s^2 ds}. \tag{5.72}$$

When $\Pi > 0$ we have

$$H_0 = \frac{\exp\left(-\int_0^T r_s ds\right)}{D_T}\Big(\left(S_0 D_0 - \widetilde{A}_T - KD_T\right)\left(N(z_1) + N(-z_2)\right) \\ -\frac{1}{2}\left(B - \sqrt{\Pi}\right)\rho(z_1) + \frac{1}{2}\left(B + \sqrt{\Pi}\right)\rho(z_2)\Big), \tag{5.73}$$

where

$$z_1 = \frac{-B - \sqrt{\Pi}}{2C}, \qquad z_2 = \frac{-B + \sqrt{\Pi}}{2C}. \tag{5.74}$$

A calculation then shows that the delta in this case is given by:

$$\frac{\partial H_0}{\partial S_0} = \frac{D_0 \exp\left(-\int_0^T r_s ds\right)}{D_T} \Big(N(z_1) + N(-z_2)\Big). \tag{5.75}$$

It is worth mentioning here that the polynomial $\mathcal{P}_1(z)$ considered above is guaranteed to be fully quadratic; the coefficient C will never be zero, since this would imply that $\beta_s = 0$ and this would reduce the expansion of Y_∞ to a first chaos.

5.7 Combination of factorisable second chaos models driven by a single Brownian motion

Now we consider the case in which both generators admit a second factorisable chaos expansion. The two generators are:

$$Y_\infty = \int_0^\infty \alpha_s dW_s + \int_0^\infty \beta_s \int_0^s \gamma_{s_1} dW_{s_1} dW_s, \tag{5.76}$$

and

$$X_\infty = \int_0^\infty \theta_s dW_s + \int_0^\infty \epsilon_s \int_0^s \zeta_{s_1} dW_{s_1} dW_s. \tag{5.77}$$

for some deterministic functions of one variable $\{\alpha_s, \beta_s, \gamma_s, \theta_s, \epsilon, \zeta_s\}$. We consider here the case of a 'one–factor' model, where X_∞ and Y_∞ are based on the same Brownian motion $\{W_t\}$. In what follows, in order to keep the notation compact, and to have suggestive expressions for the state variables involved, we make use of the functions A_t, B_t and C_t appearing in (5.57), and in addition we define the following deterministic processes:

$$\begin{aligned} \Theta_t &= \int_t^\infty \left(\theta_s^2 + \epsilon_s^2 \widetilde{Q}_s\right) ds, \qquad E_t = 2\int_t^\infty \theta_s \epsilon_s ds, \\ F_t &= \int_t^\infty \epsilon_s^2 ds. \end{aligned} \tag{5.78}$$

We define the compensators:

$$Q_t = \int_0^t \gamma_s^2 ds, \qquad \widetilde{Q}_t = \int_0^t \zeta_s^2 ds, \tag{5.79}$$

for the random processes:

$$R_t = \int_0^t \gamma_s dW_s, \qquad \widetilde{R}_t = \int_0^t \zeta_s dW_s, \tag{5.80}$$

respectively. Then for the asset price dynamics we have:

$$S_t = \frac{A_t + B_t R_t + C_t \left(R_t^2 - Q_t\right)}{\Theta_t + E_t \widetilde{R}_t + F_t \left(\widetilde{R}_t^2 - \widetilde{Q}_t\right)}. \tag{5.81}$$

The short rate in this model is given by

$$r_t = \frac{\left(\theta_t + \epsilon_t \widetilde{R}_t\right)^2}{\Theta_t + E_t \widetilde{R}_t + F_t \left(\widetilde{R}_t^2 - \widetilde{Q}_t\right)}, \tag{5.82}$$

and the dividend rate process for the asset price process $\{S_t\}$ is

$$\delta_t = \frac{\left(\alpha_t + \beta_t R_t\right)^2}{A_t + B_t R_t + C_t \left(R_t^2 - Q_t\right)}. \tag{5.83}$$

In contrast to the considerations of previous sections, we have to deal with stochastic interest rates. This is an interesting feature of the present example. In addition the volatility is also stochastic, as we shall show shortly. Given these facts, it is legitimate to consider this model as a starting point for the pricing of 'hybrid' products.

The key result here is that in the expressions above the *only state variables are the Gaussian random variables R_t and $\widetilde{R}_t$.*

For the volatility of the asset we have:

$$\sigma_t = \frac{\gamma_t B_t + 2\gamma_t C_t R_t}{A_t + B_t R_t + C_t \left(R_t^2 - Q_t\right)} - \frac{\zeta_t E_t + 2\zeta_t F_t \widetilde{R}_t}{\Theta_t + E_t \widetilde{R}_t + F_t \left(\widetilde{R}_t^2 - \widetilde{Q}_t\right)}, \tag{5.84}$$

and the market price of risk takes the form:

$$\lambda_t = -\frac{\zeta_t E_t + 2\zeta_t F_t \widetilde{R}_t}{\Theta_t + E_t \widetilde{R}_t + F_t \left(\widetilde{R}_t^2 - \widetilde{Q}_t\right)}. \tag{5.85}$$

The same formulae hold in the case of a foreign exchange model where $S_t = S_t^{ij}$ is the price of one unit of currency i in units of currency j, where

$$r_t^i = \delta_t = \frac{(\alpha_t + \beta_t R_t)^2}{A_t + B_t R_t + C_t (R_t^2 - Q_t)}, \tag{5.86}$$

and

$$r_t^j = r_t = \frac{\left(\theta_t + \epsilon_t \widetilde{R}_t\right)^2}{\Theta_t + E_t \widetilde{R}_t + F_t \left(\widetilde{R}_t^2 - \widetilde{Q}_t\right)}. \tag{5.87}$$

Now, we see that all the formulae above can be expressed in terms of two state variables which we denote by

$$Z_t = \frac{R_t}{\sqrt{Q_t}}, \qquad \widetilde{Z}_t = \frac{\widetilde{R}_t}{\sqrt{\widetilde{Q}_t}}. \tag{5.88}$$

It is straightforward to see that Z_t and $\widetilde{Z}_t$ are both standard $N(0,1)$ random variables. In terms of these variables the asset price is:

$$S_t = \frac{A_t + B_t\sqrt{Q_t}Z_t + C_t Q_t (Z_t^2 - 1)}{\Theta_t + E_t\sqrt{\widetilde{Q}_t}\widetilde{Z}_t + F_t\widetilde{Q}_t\left(\widetilde{Z}_t^2 - 1\right)}. \tag{5.89}$$

The interest rate is given by:

$$r_t = \frac{\left(\theta_t + \epsilon_t\sqrt{\widetilde{Q}_t}\widetilde{Z}_t\right)^2}{\Theta_t + E_t\sqrt{\widetilde{Q}_t}\widetilde{Z}_t + F_t\sqrt{\widetilde{Q}_t}\left(\widetilde{Z}_t^2 - 1\right)}, \tag{5.90}$$

and the dividend rate is given by:

$$\delta_t = \frac{(\alpha_t + \beta_t\sqrt{Q_t}Z_t)^2}{A_t + B_t\sqrt{Q_t}Z_t + C_t Q_t (Z_t^2 - 1)}. \tag{5.91}$$

The volatility is given in terms of Z_t and $\widetilde{Z}_t$ by:

$$\sigma_t = \frac{\gamma_t B_t + 2\gamma_t C_t\sqrt{Q_t}Z_t}{A_t + B_t\sqrt{Q_t}Z_t + C_t Q_T (Z_t^2 - 1)} - \frac{\zeta_t E_t + 2\zeta_t F_t\sqrt{\widetilde{Q}_t}\widetilde{Z}_t}{\Theta_t + E_t\sqrt{\widetilde{Q}_t}\widetilde{Z}_t + F_t\widetilde{Q}_t\left(\widetilde{Z}_t^2 - 1\right)}. \tag{5.92}$$

Finally the market price of risk vector takes the form

$$\lambda_t = -\frac{\zeta_t E_t + 2\zeta_t F_t\sqrt{\widetilde{Q}_t}\widetilde{Z}_t}{\Theta_t + E_t\sqrt{\widetilde{Q}_t}\widetilde{Z}_t + F_t\widetilde{Q}_t\left(\widetilde{Z}_t^2 - 1\right)}. \tag{5.93}$$

We note that under a foreign exchange environment we have the following interpretation:

$$r_t^i = \delta_t, \qquad r_t^j = r_t, \qquad \sigma_t^{ij} = \sigma_t, \tag{5.94}$$

and

$$\lambda_t^i = \frac{\zeta_t E_t + 2\zeta_t F_t \sqrt{\widetilde{Q}_t} \widetilde{Z}_t}{\Theta_t + E_t \sqrt{\widetilde{Q}_t} \widetilde{Z}_t + F_t \widetilde{Q}_t \left(\widetilde{Z}_t^2 - 1 \right)}, \tag{5.95}$$

$$\lambda_t^j = \frac{\gamma_t B_t + 2\gamma_t C_t \sqrt{Q_t} Z_t}{A_t + B_t \sqrt{Q_t} Z_t + C_t Q_T \left(Z_t^2 - 1 \right)}. \tag{5.96}$$

Let us now consider the pricing of European derivatives in this model. We provide the relevant calculations for a European call option. Consider the payoff

$$H_T = (S_T - K)^+. \tag{5.97}$$

The pricing formula as usual takes the form

$$H_0 = E\left[\frac{V_T}{V_0} (S_t - K)^+ \right], \tag{5.98}$$

where the expectation is taken under the natural measure P. Now, we already know the pricing kernel which is given by the relation

$$V_t = \Theta_t + E_t \sqrt{\widetilde{Q}_t} \widetilde{Z}_t + F_t \widetilde{Q}_t \left(\widetilde{Z}_t^2 - 1 \right). \tag{5.99}$$

It should be clear that one has to consider the sign of the expression $V_T(S_T - K)$ which appears inside the expectation (5.98) in order to be able to determine the limits of the corresponding integral. We note that expression (5.98) translates into

$$H_0 = \frac{1}{V_0} E\left[\left(\mathcal{P}_1(Z_T) - K\mathcal{P}_2(\widetilde{Z}_T) \right)^+ \right], \tag{5.100}$$

where the polynomials $\mathcal{P}_1(z)$ and $\mathcal{P}_2(z)$ are defined by

$$\mathcal{P}_1(z) \quad = \quad A_T + B_T \sqrt{Q_T} z + C_T Q_T (z^2 - 1), \tag{5.101}$$

and

$$\mathcal{P}_2(z) \quad = \quad \Theta_T + E_T\sqrt{\widetilde{Q}_T}z + F_T\widetilde{Q}_T(z^2-1). \tag{5.102}$$

Then the option price is given by

$$H_0 = \frac{1}{\Theta_0}\int_{-\infty}^{\infty}\int_{z^*}^{\infty}\Big(\mathcal{P}_1(z) - K\mathcal{P}_2(\widetilde{z})\Big)f(z,\widetilde{z})dz\,d\widetilde{z}, \tag{5.103}$$

with $f(z,\widetilde{z})$ being the bivariate standard normal density function, and z^* is the critical value of z, given as a function of $\widetilde{z}$, that ensures the integral is positive.

The general discussion above can be put into a more specific form if we consider a certain reduction of the generality of the model. In particular, for the rest of this section, we consider the case for which the deterministic functions γ_s and ζ_s are equal. This assumption will play an important role in the calculation of the expectation in (5.100). To be direct, we now assume $\gamma_t = \zeta_t$ and hence that $Z_T = \widetilde{Z}_T = X_T$ and $Q_T = \widetilde{Q}_T$. Now we define the following deterministic quantities:

$$\begin{aligned} A &= (A_T - C_TQ_T) - K\left(\Theta_T - F_TQ_T\right) \\ B &= \sqrt{Q_T}\left(B_T - KE_T\right) \\ C &= Q_T\left(C_T - KF_T\right), \end{aligned} \tag{5.104}$$

and note that the formula under consideration involves a function of one variable only, that is to say, we have:

$$V_T\left(S_T - K\right) = A + BX_T + CX_T^2. \tag{5.105}$$

It now becomes clear why we made the assumption $\gamma_t = \zeta_t$. With this assumption we only have to consider a univariate standard normal distribution, and the corresponding results become explicit. We therefore need to consider the sign of the polynomial

$$\mathcal{P}(x) = A + Bx + Cx^2. \tag{5.106}$$

For this purpose we also note the discriminant $\Delta = B^2 - 4AC$.

Case 1: $\mathbf{C = 0}$, and $\mathbf{B > 0}$. Then

$$H_0 = \frac{1}{\Theta_0}\left(AN(-x_0) + B\rho(x_0)\right), \tag{5.107}$$

where $N(x)$ is the cumulative standard normal distribution, and $\rho(x)$ the associated density function. The value $x_0 = -A/B$ is the only root of the polynomial under consideration.

Case 2: $\mathbf{C = 0}$ and $\mathbf{B < 0}$. Then we have

$$H_0 = \frac{1}{\Theta_0}\left(AN(x_0) - B\rho(x_0)\right). \tag{5.108}$$

Case 3: $\mathbf{C > 0}$ and $\mathbf{\Delta > 0}$. In this case we have two roots

$$x_1 = \frac{-B - \sqrt{\Delta}}{2A}, \qquad \text{and} \qquad x_2 = \frac{-B + \sqrt{\Delta}}{2A}, \tag{5.109}$$

and the option price is

$$\begin{aligned} H_0 &= \frac{1}{\Theta_0}\left(A_T - K\Theta_T\right)\left(N(x_1) + N(-x_2)\right) \\ &- \frac{1}{2\Theta_0}\left(B - \sqrt{\Delta}\right)\rho(x_1) + \frac{1}{2\Theta_0}\left(B + \sqrt{\Delta}\right)\rho(x_2). \end{aligned} \tag{5.110}$$

Case 4: $\mathbf{C < 0}$ and $\mathbf{\Delta > 0}$. Here the analysis is similar, and we get

$$\begin{aligned} H_0 &= \frac{1}{\Theta_0}\left(A_T - K\Theta_T\right)\left(N(x_1) - N(x_2)\right) \\ &- \frac{1}{2\Theta_0}\left(B - \sqrt{\Delta}\right)\rho(x_1) + \frac{1}{2\Theta_0}\left(B + \sqrt{\Delta}\right)\rho(x_2). \end{aligned} \tag{5.111}$$

The case of negative or zero discriminant gives rise to a degenerate situation in which the option is either guaranteed to expire in the money (when $C > 0$), and we have $H_0 = A + C$, or the option will expire out of the money (when $C < 0$), in which case $H_0 = 0$.

5.8 Geometric Brownian motion revisited

In this section we show how to generate the well–known constant parameters–geometric Brownian motion model for asset price dynamics. Here we take a somewhat different point of view: instead of considering a specific chaos expansion for the two generators involved, as in previous sections, we shall investigate the connection between the geometric Brownian motion model and the associated pricing kernels. Therefore, here we do not use the Wiener chaos representation for the pricing kernel processes, but rather make use of the generic machinery in the form

$$S_t = \frac{\text{Var}_t\left[Y_\infty\right]}{\text{Var}_t\left[X_\infty\right]}. \tag{5.112}$$

In a more general context, it is worth mentioning that one is not *obliged* to make use of the Wiener chaos expansion technique. This technique is used to derive explicit results, and to parametrise the models in a systematic way. However, if we are able in some other way to generate the same results, then this method is welcome as well. In the present context we are looking for a representation of the two generators X_∞ and Y_∞, in the case of an asset following a geometric Brownian motion process. This turns out to be a straightforward calculation, once we recall the definition of X_∞. This involves the specification of the short rate and the pricing kernel process, and in the one dimensional case takes the form

$$X_\infty = \int_0^\infty \sqrt{r_t V_t} dW_t. \tag{5.113}$$

In the Black–Scholes economy we have a constant interest rate r and a constant market price of risk λ. The equation for the pricing kernel reads

$$dV_t = -rV_t dt - \lambda V_t dW_t. \tag{5.114}$$

The solution to this equation is

$$V_t = V_0 \exp\left(-rt - \lambda W_t - \frac{1}{2}\lambda^2 t\right). \tag{5.115}$$

As a consequence, we have

$$
\begin{aligned}
X_\infty &= \int_0^\infty \sqrt{r_t V_t} dW_t \\
&= \sqrt{rV_0} \int_0^\infty \exp\left(-\frac{1}{2}rt - \frac{1}{2}\lambda W_t - \frac{1}{4}\lambda^2 t\right) dW_t.
\end{aligned} \tag{5.116}
$$

It is an exercise to check that

$$
V_t = \mathbb{E}_t\left[X_\infty^2\right] - \left(\mathbb{E}_t\left[X_\infty\right]\right)^2, \tag{5.117}
$$

as it of course should. In particular, we note that

$$
E_t[X_\infty^2] = rV_0 E_t\left[\left(\int_0^\infty \exp\left(-\frac{1}{2}rs - \frac{1}{2}\lambda W_s - \frac{1}{4}\lambda^2 s\right) dW_s\right)^2\right], \tag{5.118}
$$

and thus

$$
\begin{aligned}
E_t[X_\infty^2] &= rV_0 \left(\int_0^t \exp\left(-\frac{1}{2}rs - \frac{1}{2}\lambda W_s - \frac{1}{4}\lambda^2 s\right) dW_s\right)^2 \\
&+ rV_0 E_t\left[\int_t^\infty \exp\left(-rs - \lambda W_s - \frac{1}{2}\lambda^2 s\right) ds\right],
\end{aligned} \tag{5.119}
$$

by use of the Itô isometry. On the other hand, we note that

$$
\begin{aligned}
&E_t\left[\int_t^\infty \exp\left(-rs - \lambda W_s - \frac{1}{2}\lambda^2 s\right) ds\right] \\
&= \int_t^\infty E_t\left[\exp\left(-rs - \lambda W_s - \frac{1}{2}\lambda^2 s\right)\right] ds \\
&= \exp\left(-\lambda W_t - \frac{1}{2}\lambda^2 t\right) \int_t^\infty \exp\left(-rs\right) ds \\
&= \frac{1}{r}\exp\left(-rt - \lambda W_t - \frac{1}{2}\lambda^2 t\right).
\end{aligned} \tag{5.120}
$$

Now making use of the fact that

$$
E_t[X_\infty] = \sqrt{rV_0} \int_0^t \exp\left(-\frac{1}{2}rs - \frac{1}{2}\lambda W_s - \frac{1}{4}\lambda^2 s\right) dW_s, \tag{5.121}
$$

we see that (5.117) implies (5.115).

Finally, we need to check that axioms (A2) and (A3), are satisfied in this model. Thus we need to verify that the process $\{N_t\}$ defined by

$$N_t = V_t + \int_0^t r_s V_s ds, \tag{5.122}$$

is a martingale. Now, more explicitly we have

$$\begin{aligned} N_t &= V_0 \exp\left(-rt - \lambda W_t - \frac{1}{2}\lambda^2 t\right) \\ &+ rV_0 \int_0^t \exp\left(-rs - \lambda W_s - \frac{1}{2}\lambda^2 s\right) ds. \end{aligned} \tag{5.123}$$

Thus, we have:

$$\begin{aligned} E_t[N_T] &= V_0 \exp(-rT) \exp\left(-\lambda W_t - \frac{1}{2}\lambda^2 t\right) \\ &+ rV_0 E_t\left[\int_0^T \exp\left(-rs - \lambda W_s - \frac{1}{2}\lambda^2 s\right) ds\right] \\ &= V_0 \exp(-rT) \exp\left(-\lambda W_t - \frac{1}{2}\lambda^2 t\right) \\ &+ rV_0 \int_0^t \exp\left(-rs - \lambda W_s - \frac{1}{2}\lambda^2 s\right) ds \\ &+ rV_0 \int_t^T E_t\left[\exp\left(-rs - \lambda W_s - \frac{1}{2}\lambda^2 s\right)\right] ds \\ &= V_0 \exp(-rT) \exp\left(-\lambda W_t - \frac{1}{2}\lambda^2 t\right) + \int_0^t r_s V_s ds \\ &+ rV_0 \left(\int_t^T \exp(-rs)\, ds\right) \exp\left(-\lambda W_t - \frac{1}{2}\lambda^2 t\right) \\ &= V_t + \int_0^t r_s V_s ds \\ &= N_t, \end{aligned} \tag{5.124}$$

and that shows that axiom (A2) is satisfied.

Now we introduce a second pricing kernel Ψ_t based on the same Brownian motion, satisfying

$$d\Psi_t = -f\Psi_t dt - \gamma \Psi_t dW_t. \tag{5.125}$$

This process can be interpreted as the pricing kernel appearing when the risky asset with price process $\{S_t\}$ is taken to be numeraire. We interpret f as the dividend rate or foreign interest rate associated with $\{S_t\}$, and γ as the associated market price of risk for asset priced in units of $\{S_t\}$. For Ψ_t we obtain:

$$\Psi_t = \Psi_0 \exp\left(-ft - \gamma W_t - \frac{1}{2}\gamma^2 t\right), \tag{5.126}$$

and for the associated random variable Y_∞ we have

$$Y_\infty = \sqrt{f\Psi_0}\int_0^\infty \exp\left(-\frac{1}{2}ft - \frac{1}{2}\gamma W_t - \frac{1}{4}\gamma^2 t\right) dW_t. \tag{5.127}$$

Clearly, this expression for Y_∞ leads back to the given expression for Ψ_t above. Finally, for the process $\{S_t\}$ we deduce that

$$\begin{aligned} S_t &= \frac{\mathrm{Var}_t\left[Y_\infty\right]}{\mathrm{Var}_t\left[X_\infty\right]} \\ &= \frac{\Psi_t}{V_t} \\ &= \frac{\Psi_0 \exp\left(-ft - \gamma W_t - \frac{1}{2}\gamma^2 t\right)}{V_0 \exp\left(-rt - \lambda W_t - \frac{1}{2}\lambda^2 t\right)} \\ &= S_0 \exp\left((r-f)t + \lambda\sigma t + \sigma W_t - \frac{1}{2}\sigma^2 t\right), \end{aligned} \tag{5.128}$$

where the initial value of the asset is

$$S_0 = \frac{\Psi_0}{V_0}, \tag{5.129}$$

and the volatility parameter is given by

$$\sigma = \lambda - \gamma. \tag{5.130}$$

Thus we recover the standard geometric Brownian motion model for an asset paying dividends at the rate f, where σ is the volatility. We made use of the fact that the asset can be expressed as the quotient of two supermartingales, which here represent the pricing kernels for the two different numeraire systems. In particular, we note that the volatility of the asset price arises as the difference of the two risk premiums. We note that the example above, based on a single Brownian factor, with constant parameters, can be generalised to encompass wider classes of models; for example models with deterministic time-dependent $r(t)$, $f(t)$, $\lambda(t)$, $\gamma(t)$, and $\sigma(t)$.

Now let us verify that the price process $\{S_t\}$ that we have constructed actually satisfies axiom (A2). That is to say, we want to show that the process $\{M_t\}$ defined by

$$M_t = V_t S_t + \int_0^t V_s D_s ds, \tag{5.131}$$

is a martingale, where $\{D_t\}$ is the divided flow process. Now, more explicitly, we have

$$M_t = \Psi_t + f \int_0^t \Psi_s ds. \tag{5.132}$$

However, it should be apparent that $\{M_t\}$ is in fact the floating–rate note martingale associated with the pricing kernel $\{\Psi_t\}$. Indeed, the same argument used earlier to show that the process $\{N_t\}$ associated with $\{V_t\}$ is a martingale in the geometric Brownian motion model is here sufficient to establish that $\{M_t\}$ is a martingale.

In summary, we see that the standard geometric Brownian motion model for an asset that pays a positive dividend stream at a constant rate satisfies all conditions of our axiomatic scheme.

5.9 General remarks and future work

In this chapter we made use of the conditional variance representation for the pricing kernel process, in the general assets framework. First, we demonstrated how the framework works in a foreign exchange setting, where there are two or more economies, and we therefore have a system of 'foreign' currencies. Then, we were able to show that this translates to a generic expression, not only for the foreign exchange case, but also to the case of a single–currency economy, the ultimate result being a formula for an asset of limited liability, along with explicit, generic results for the short rate and the dividend rate processes, and for the market price of risk vector. The framework provided here is such that the generation of models is essentially boiled down to the specification of chaotic expansions for square–integrable random variables. All explicit results for the variables involved, are easily formulated once the two Wiener chaos expansions have been posed, the starting point being equation (5.13). Moreover, for the most elementary expansions, i.e. the first and second chaos expansions along with the coherent expansion, we have to our disposal explicit expressions for the associated interest rates and risk premium vectors from the analysis in chapters 3 and 4. Based on these ideas, we were able to provide specific examples, along with option pricing formulae. It should be evident that despite the wealth of examples we do not exhaust all possibilities; in fact, there are many more models that can be generated within this framework. The choice of the models presented here is based on tractability and simplicity. No attempt is made here to provide any empirical analysis of these models. Neither we have made the most of each model as far as option pricing is concerned. Indeed, the valuation of derivatives presented here should be considered as a demonstration only, restricted to the case of European call options. There is a good deal of flexibility as far as the variety of contracts that we can consider is concerned, and this is another possible direction of future work.

Returning to the discussion of empirical studies of these models, we now discuss briefly how and why such studies could be of importance to financial institutions. One of the problems that practitioners have to face on their day–to–day routine, is the so called dimensionality problem. The valuation of 'hybrid' products is concerned with objects that depend on more than one state variables. Consider, for example, a contingent claim that is written on an asset with price process $\{S_t\}$ and on the overnight rate with process $\{r_t\}$. Or we can envisage a derivative written on the foreign exchange between two countries and on the overnight rates of both currencies. For such a family of problems, one is faced with the task of providing a model for each of the state variables, that is to say, one needs to specify the dynamics for both assets and interest rates. To our knowledge, there is no known framework that will systematically provide such a solution, a way to rationally provide the dynamics of all state variables on a coherent basis. What is common in practice, is an *a priori* assumption for the dynamics of the asset, and another model for the dynamics of the overnight rate. However, there is typically no natural or intuitive connection between these models. In the best case, considering some correlation between the different Wiener processes driving the assets and interest rates is what is offered in terms of modelling in such situations according to market practice. This discussion generalises to the case of stochastic volatility, and to the evaluation of more complicated positions, such as volatility products, or hybrid products written on assets, volatilities, and rates. Some of these contingent claims (volatility swaps, swaptions etc.) are currently traded on an over–the–counter basis; however, the most complicated ones are not traded as yet. Nevertheless, there is every reason to believe that in the future these types of derivatives will be introduced. In practice one is forced to consider simple deterministic evolution for the interest rates, in order to be able to provide a framework for efficient and fast valuation of prices and hedging parameters. Although this is common use in many cases, it is important to provide alternatives. For sure, this is useless when one is pricing pure interest rate derivatives, and practitioners use stochastic dynamics for interest rates in this case, which is easier to model since the only underlying assets are the discount bonds and the money market account. However, the very nature of interest rates and discount bonds, as the main tools of connecting different points in time and quantifying the notion of time value, is why they appear in the valuation of other types of derivatives. Although the modern theory of asset pricing provides the flexibility of using any asset in the market as numeraire (cf. El Karoui et al 1995), the common practise is to use the money market account as such, and transfer to the so–called risk–neutral probability measure, under which non–dividend paying asset price processes become martingales. This is the essence of the monumental work by Harrison and Kreps (1979), and Harrison and Pliska

(1982). The point for our discussion is that in the risk–neutral measure what usually appears as a 'discount factor' when pricing derivatives, is the quotient of the money market account at two different points in time. Therefore, as one would expect intuitively, interest rates, or equivalently discount bonds, are directly involved in the pricing of any derivative, and it can only be conceived as a defeat of modellers and practitioners that in this case we should think of rates as being deterministic, or even constant. These considerations are of importance to practitioners, who need a 'good', intuitive model that will provide a way to generate dynamics for all variables under the same general assumptions, and give the flexibility for efficient numerical calculations.

The contents of this chapter provide a viable solution to the considerations above, and show a way forward for further progress in the matter. First, we are able to construct a mechanism that generates a large variety of models for assets, interest rates, and volatilities, *all in the same general framework*. This is because the considerations in chapters 3 and 4 for term structure modelling are generalised to other assets as well. The first two sections of this chapter show how to do this, and we derive the basic equations (5.4), (5.5) and (5.13). The chaotic framework provides a possible way around the 'curse of dimensionality', in the sense that once we specify a chaos expansion for each of the underlying random variables, we are left with a specific economic framework that provides dynamics for all assets involved, let them be stocks, interest rates, volatilities or foreign exchanges. Intuitively speaking, a market in which all the relevant stochastic variables are connected in many different ways is more realistic. In addition, the simple cases of deterministic models for variables such as short rates are incorporated within the framework as a subset. It is legitimate therefore to regard the chaotic approach as a tool that provides extra degrees of freedom to financial modelling if and when they are needed. The degree of complexity one has to deal with is entirely in one's own hands. The main idea when it comes to modelling in this case, is that one inverts the argument of the conditional variance representation and Wiener chaos, and considers the deterministic chaos coefficients as *exogenously specified*. Then, it is a matter of mathematical taste, analytical tractability, and numerical efficiency, that determines which expansions one is going to use. It should be clarified that, although the word expansion is used in this project, and indeed we make use of the chaotic expansion on Wiener space, there is no 'asymptotic' result here. In other words, we do not *approximate*, but instead we infer specific models. Or, to consider the matter from a different perspective, the only form of approximation is an approximation to a specific model, which in principle would be superior to all others, and it comes as a consequence of considering the whole expansion. One cannot help, yet one more time, to make the parallelism with theoretical physics: all theories so far in this direction tend to be considered approximations of a general 'theory of everything'. Coming back to the previous considerations, we forge ahead with the question of stochastic volatility, and stochastic interest rates. The first few models considered in this chapter, were the ones that had a first chaos expansion for the denominator of equation (5.13). This translates to deterministic interest rates, though in the foreign exchange setting the interest rates of the 'foreign' economy are determined by the expansion of the numerator, and are always stochastic. The same holds for the volatility: the simple models include deterministic volatility. However, as we progress to more sophisticated models, randomness presents itself in many faces, and this includes stochastic volatility and short rates. This situation is what most would now call a *local* volatility model, since we are using a one dimensional Brownian motion. In particular, we direct the discussion to the combination of second chaos case, which is described in section 5.7. What we are dealing with here is a model that exhibits stochasticity of all variables, in particular stochastic interest rates and stochastic asset dynamics, and at the same time offers considerable flexibility, in the sense that we end up with Gaussian expressions. Again here, stochastic should be interpreted as local, for both volatility and interest rates, because of the use of a one factor model. In view of this attractive feature, it is reasonable to explore this specific model further. Indeed, a future area of research is the empirical investigation of the viability of this model, and the use of it in real situations.

A word on more sophisticated models, such as models with two or more factors. These models will exhibit a 'real' stochastic volatility, and will enable us to consider a unified framework for stochastic volatility and interest rate modelling. We colonise this area in the next chapter, where we deal with the same situation as in the last example of this section, but in the two dimensional case.

Chapter 6

Unified theories of stochastic volatility and interest rates

6.1 Stochastic volatility models

In this chapter we continue our investigation of the framework for general asset dynamics. What differentiates the arguments presented here with those of the preceding chapter is the construction of a stochastic environment for volatilities, based on a market driven by two factors. In the previous chapter, most of the examples we have considered evolved in a market driven by a one–dimensional Wiener process. The result was stochastic evolution for volatilities and rates, but perfectly correlated with the evolution of assets, that is to say, what we have essentially encountered were *local* volatility models (see, e.g., Dupire, 1994). It is only when we enrich the random background of our framework with a second independent Wiener process, that one can talk about stochastic volatility in its whole generality. For a general discussion on stochastic volatility, and conditions on how to construct a complete market in this case, by the use of exchange–traded options as non–redundant assets, see Davis (2004). For the rest of this chapter we shall have a two–dimensional Brownian factor driving the market. We present here one such model that is general enough to demonstrate most of the relevant theoretical considerations, and at the same time is tractable in the sense of explicit results for standard options being obtainable.

6.2 Two factor second chaos term structure

We begin by developing a term structure model that we shall call a two–factor second chaos model. This model uses a second Wiener chaos expansion in two dimensions, and generalises the one–factor model considered in chapter 5. Consider the generator Y_∞ expanded under the following scheme:

$$Y_\infty = \int_0^\infty \phi_s^\alpha dW_s^\alpha + \int_0^\infty \int_0^s \phi_{ss_1}^{\alpha\beta} dW_{s_1}^\alpha dW_s^\beta, \qquad \alpha, \beta = 1, 2 \qquad . \tag{6.1}$$

In the representation above we use the Einstein convention; that is, there is an implied summation for both factors α and β. As a result here there are six exogenously specified coefficient functions which we summarise as follows:

$$\begin{aligned} &\phi_s^1, \quad \phi_s^2 \qquad \text{(first chaos)} \\ &\phi_{ss_1}^{11}, \quad \phi_{ss_1}^{12}, \phi_{ss_1}^{21}, \phi_{ss_1}^{22} \qquad \text{(second chaos)}. \end{aligned} \tag{6.2}$$

One is naturally lead to the question of the conditional variance of the above generator. This calculation becomes a little more complicated. It is possible nevertheless to write down the pricing kernel for this model. We omit the details of the calculations, and give the following result:

$$V_t = \sum_{i=1}^{6} V_t^i, \tag{6.3}$$

where we define:

$$\begin{aligned} V_t^1 &= \int_t^\infty \left(\phi_s^1 + \int_0^t \phi_{ss_1}^{11} dW_{s_1}^1 \right)^2 ds + \int_t^\infty \int_t^s \left(\phi_{ss_1}^{11}\right)^2 ds_1 ds \\ V_t^2 &= \int_t^\infty \left(\phi_s^2 + \int_0^t \phi_{ss_1}^{22} dW_{s_1}^2 \right)^2 ds + \int_t^\infty \int_t^s \left(\phi_{ss_1}^{22}\right)^2 ds_1 ds \\ V_t^3 &= \int_t^\infty \left(\left(\int_0^t \phi_{ss_1}^{21} dW_{s_1}^2 \right)^2 + \int_t^s \left(\phi_{ss_1}^{21}\right)^2 ds_1 \right) ds \end{aligned}$$

$$
\begin{aligned}
V_t^4 &= \int_t^\infty \left(\left(\int_0^t \phi^{12}_{ss_1} dW^1_{s_1} \right)^2 + \int_t^s \left(\phi^{12}_{ss_1}\right)^2 ds_1 \right) ds \\
V_t^5 &= 2\int_t^\infty \phi^1_s \int_0^t \phi^{21}_{ss_1} dW^2_{s_1} ds + 2\int_t^\infty \int_0^t \phi^{11}_{ss_1} dW^1_{s_1} \int_0^t \phi^{21}_{ss_1} dW^2_{s_1} ds \\
V_t^6 &= 2\int_t^\infty \phi^2_s \int_0^t \phi^{12}_{ss_1} dW^1_{s_1} ds + 2\int_t^\infty \int_0^t \phi^{22}_{ss_1} dW^2_{s_1} \int_0^t \phi^{12}_{ss_1} dW^1_{s_1} ds. \quad (6.4)
\end{aligned}
$$

6.3 Factorisable generalised second chaos

It is in the interest of clarity, simplicity and tractability if we consider the relevant factorisable case, in the spirit of chapter 5. The degrees of freedom in this case are ten deterministic functions of one variable, with the condition that all are square–integrable. We introduce the following parametrisation:

$$
\begin{aligned}
\phi^1_s &= \delta_s, \qquad \phi^2_s = \alpha_s \qquad \phi^{11}_{ss_1} = \epsilon_s \zeta_{s_1}, \\
\phi^{22}_{ss_1} &= \beta_s \gamma_{s_1}, \qquad \phi^{21}_{ss_1} = \eta_s \theta_{s_1}, \qquad \phi^{12}_{ss_1} = \psi_s k_{s_1}. \qquad (6.5)
\end{aligned}
$$

As was the case in previous chapters, the advantage of the factorisability of a chaos model is that state variables become more explicit. In the second chaos case we have Gaussian martingales as the only stochastic terms. The pricing kernel for the model here is given by

$$
\begin{aligned}
V_t &= D_t + E_t L_t + F_t \left(L_t^2 - N_t\right) + A_t + B_t R_t + C_t \left(R_t^2 - Q_t\right) \\
&+ 2\Theta_t \left(\Delta_t + L_t M_t\right) + 2K_t \left(\Phi_t + R_t \Xi_t\right) \\
&+ H_t \left(\Theta_t^2 - \Lambda_t\right) + Y_t + \Psi_t \left(K_t^2 - \Sigma_t\right) + \Gamma_t. \qquad (6.6)
\end{aligned}
$$

The only stochastic terms are the following martingales:

$$
\begin{aligned}
L_t = \int_0^t \zeta_s dW^1_s, \qquad K_t = \int_0^t k_s dW^1_s, \\
R_t = \int_0^t \gamma_s dW^2_s, \qquad \Theta_t = \int_0^t \theta_s dW^2_s. \qquad (6.7)
\end{aligned}
$$

Additionally, we define the following deterministic quantities:

$$
D_t = \int_t^\infty \left(\delta_s^2 + \epsilon_s^2 N_s\right) ds, \qquad B_t = 2\int_t^\infty \alpha_s \beta_s ds, \qquad N_t = \int_0^t \zeta_s^2 ds,
$$

$$
\begin{aligned}
C_t = \int_t^\infty \beta_s^2 ds, \qquad & E_t = 2\int_t^\infty \delta_s \epsilon_s ds, \qquad & \Delta_t = \int_t^\infty \delta_s \eta_s ds, \\
F_t = \int_t^\infty \epsilon_s^2 ds, \qquad & M_t = \int_t^\infty \epsilon_s \eta_s ds, \qquad & \Phi_t = \int_t^\infty \alpha_s \psi_s ds, \\
A_t = \int_t^\infty \left(\alpha_s^2 + \beta_s^2 Q_s\right) ds, \qquad & Q_t = \int_0^t \gamma_s^2 ds, \qquad & \Xi_t = \int_t^\infty \beta_s \psi_s ds, \\
\Psi_t = \int_t^\infty \psi_s^2 ds, \qquad & \Gamma_t = \int_t^\infty \psi_s^2 \Sigma_s ds, \qquad & \Sigma_t = \int_0^t k_s^2 ds, \\
H_t = \int_t^\infty \eta_s^2 ds, \qquad & Y_t = \int_t^\infty \eta_s^2 \Lambda_s ds, \qquad & \Lambda_t = \int_0^t \theta_s^2 ds.
\end{aligned} \tag{6.8}
$$

The one–dimensional case is recovered when we consider the coefficients ϵ_s, ζ_s and δ_s to be non–zero, and all other coefficients to be zero. This is compatible with the parametrisation in (6.5). In this case we recover expression (4.41).

Returning to the two–dimensional case, one can write explicit expressions for the discount bonds, the short rate process, and the market price of risk vector. For the discount bond system, it is a matter of applying the formula

$$
P_{tT} = \frac{E_t[V_T]}{V_t}. \tag{6.9}
$$

On the other hand, applying Itô calculus for the process $\{V_t\}$ and by using the general expression

$$
dV_t = -r_t V_t dt - V_t \lambda_t^\alpha dW_t^\alpha, \qquad \alpha = 1, 2\,, \tag{6.10}
$$

we obtain

$$
\begin{aligned}
r_t &= V_t^{-1}\Big[(\alpha_t + \beta_t R_t)^2 + (\delta_t + \epsilon_t L_t)^2 \\
&+ 2\left(\eta_t \delta_t \Theta_t + \eta_t \epsilon_t \Theta_t L_t + \psi_t \alpha_t K_t + \psi_t \beta_t R_t K_t\right) + \eta_t^2 \Theta_t^2 + \psi_t^2 K_t^2 \Big],
\end{aligned} \tag{6.11}
$$

for the short rate process. Another way to see this is to consider the definition of the generator Y_∞:

$$
Y_\infty = \int_0^\infty \tau_s dW_s, \tag{6.12}
$$

where, for this particular model, the two dimensional vector process $\{\tau_t\}$ takes the following form:

$$\begin{aligned}
\tau_t^1 &= \delta_t + \epsilon_t L_t + \eta_t \Theta_t, \\
\tau_t^2 &= \alpha_t + \beta_t R_t + \psi_t K_t.
\end{aligned} \tag{6.13}$$

Now recalling that by construction we have

$$\tau_t^2 = r_t V_t, \tag{6.14}$$

we can solve the equation above to recover expression (6.11). The components

$$\lambda_t^\alpha = [\lambda_t^1, \lambda_t^2], \tag{6.15}$$

of the market price of risk vector are given by:

$$\begin{aligned}
\lambda_t^1 &= -V_t^{-1}\left[\zeta_t E_t + 2\Big(\zeta_t F_t L_t + \zeta_t \Theta_t M_t + k_t\left(\Phi_t + R_t \Xi_t + K_t \Psi_t\right)\Big)\right], \\
\lambda_t^2 &= -V_t^{-1}\left[\gamma_t B_t + 2\Big(\gamma_t C_t R_t + \gamma_t K_t \Xi_t + \theta_t\left(\Delta_t + L_t M_t + \Theta_t H_t\right)\Big)\right].
\end{aligned} \tag{6.16}$$

6.4 Second chaos stochastic volatility

We now propose a model for the stochastic dynamics of an asset price process $\{S_t\}$ of limited liability, and its volatility $\{\sigma_t\}$ where there is a short rate process $\{r_t\}$ in the model, and a system of discount bonds $\{P_{tT}\}$. The model under consideration is essentially a generalisation of the combination of the second chaos model presented in section 5.7. In this model we considered second chaos expansions for the generators appearing in the numerator and denominator of equation (5.13). However, matters can be considerably extended if one develops the same arguments in a two factor model, i.e. with a two–dimensional Wiener process as the random driver of the economy. This gives rise to stochastic dynamics for the volatility process. We will demonstrate that even in the case of a two–dimensional Wiener space, some of the resulting models are tractable enough to provide explicit results for vanilla options.

Let us begin by rewriting the generic formula for asset dynamics, since it will be the starting point yet for the following considerations:

$$S_t = \frac{\mathrm{Var}_t[Y_\infty]}{\mathrm{Var}_t[X_\infty]}. \tag{6.17}$$

The two Wiener processes are independent and form the vector $\{W_t\} = \{W_t^1, W_t^2\}$. We consider the expansion (6.1) for the generator Y_∞ but in the factorisable case, i.e. we impose the factorisation (6.5). We will make use of the same terminology and notation as in the previous section for all the degrees of freedom arising from the conditional variance representation for the pricing kernel associated with Y_∞. In addition we denote by $\{\widetilde{V}_t\}$ the pricing kernel process that corresponds to the generator X_∞. The coefficients connected to this kernel will be denoted with the tilde sign, to differentiate them from the ones of the kernel process $\{V_t\}$. Thus, we have for example, the functions $\widetilde{\phi}^{12}_{ss_1}$, $\widetilde{\phi}^2_s$ and so on. Having clarified the notation we move to the factorisable case. We consider the model that arises when we factorise all of the coefficients for the generators X_∞ and Y_∞. The result is still a very general model supporting twenty functions of one variable each. We give results for asset dynamics and volatilities for this model; later on we reduce the generality further to reduce the degrees of freedom. Based on (6.17), and by virtue of (6.6), we have

$$S_t = \frac{V_t}{\widetilde{V}_t}, \tag{6.18}$$

where the process $\{\widetilde{V}_t\}$ is given by

$$\begin{aligned} \widetilde{V}_t &= \widetilde{D}_t + \widetilde{E}_t\widetilde{L}_t + \widetilde{F}_t\left(\widetilde{L}_t^2 - \widetilde{N}_t\right) + \widetilde{A}_t + \widetilde{B}_t\widetilde{R}_t + \widetilde{C}_t\left(\widetilde{R}_t^2 - \widetilde{Q}_t\right) \\ &+ 2\widetilde{\Theta}_t\left(\widetilde{\Delta}_t + \widetilde{L}_t\widetilde{M}_t\right) + 2\widetilde{K}_t\left(\widetilde{\Phi}_t + \widetilde{R}_t\widetilde{\Xi}_t\right) \\ &+ \widetilde{H}_t\left(\widetilde{\Theta}_t^2 - \widetilde{\Lambda}_t\right) + \widetilde{Y}_t + \widetilde{\Psi}_t\left(\widetilde{K}_t^2 - \widetilde{\Sigma}_t\right) + \widetilde{\Gamma}_t. \end{aligned} \tag{6.19}$$

This is the representation for the price $\{S_t\}$ in a two–dimensional Wiener environment. Recall, however, the generic relation

$$dS_t = (r_t - \delta_t + \lambda_t\sigma_t)S_t dt + \sigma_t S_t dW_t, \tag{6.20}$$

where the short rate is associated to the pricing kernel generated by X_∞, and the volatility vector is given by

$$\sigma_t = \widetilde{\lambda}_t - \lambda_t. \tag{6.21}$$

The dividend rate process $\{\delta_t\}$ is the short rate associated to the pricing kernel generated by Y_∞, whereas the risk premium vector $\{\lambda_t\}$ is the one of the pricing kernel generated by X_∞. Based on these results, we deduce the following model for the evolution of the short rate:

$$\begin{aligned} r_t &= \widetilde{V}_t^{-1}\Big[\Big(\widetilde{\alpha}_t + \widetilde{\beta}_t\widetilde{R}_t\Big)^2 + \Big(\widetilde{\delta}_t + \widetilde{\epsilon}_t\widetilde{L}_t\Big)^2 \\ &+ 2\left(\widetilde{\eta}_t\widetilde{\delta}_t\widetilde{\Theta}_t + \widetilde{\eta}_t\widetilde{\epsilon}_t\widetilde{\Theta}_t\widetilde{L}_t + \widetilde{\psi}_t\widetilde{\alpha}_t\widetilde{K}_t + \widetilde{\psi}_t\widetilde{\beta}_t\widetilde{R}_t\widetilde{K}_t\right) + \widetilde{\eta}_t^2\widetilde{\Theta}_t^2 + \widetilde{\psi}_t^2\widetilde{K}_t^2\Big]. \end{aligned} \tag{6.22}$$

As mentioned above, the volatility vector is given by the difference of the two risk premium vectors. If we represent this vector by $\{\sigma_t\} = \{\sigma_t^1, \sigma_t^2\}$ then we have the following expressions for the two coefficients:

$$\begin{aligned} \sigma_t^1 = &- \widetilde{V}_t^{-1}\left[\widetilde{\zeta}_t\widetilde{E}_t + 2\Big(\widetilde{\zeta}_t\widetilde{F}_t\widetilde{L}_t + \widetilde{\zeta}_t\widetilde{\Theta}_t\widetilde{M}_t + \widetilde{k}_t\left(\widetilde{\Phi}_t + \widetilde{R}_t\widetilde{\Xi}_t + \widetilde{K}_t\widetilde{\Psi}_t\right)\Big)\right] \\ &+ V_t^{-1}\left[\zeta_t E_t + 2\Big(\zeta_t F_t L_t + \zeta_t\Theta_t M_t + k_t\left(\Phi_t + R_t\Xi_t + K_t\Psi_t\right)\Big)\right], \end{aligned} \tag{6.23}$$

$$\begin{aligned} \sigma_t^2 = &- \widetilde{V}_t^{-1}\left[\widetilde{\gamma}_t\widetilde{B}_t + 2\Big(\widetilde{\gamma}_t\widetilde{C}_t\widetilde{R}_t + \widetilde{\gamma}_t\widetilde{K}_t\widetilde{\Xi}_t + \widetilde{\theta}_t\left(\widetilde{\Delta}_t + \widetilde{L}_t\widetilde{M}_t + \widetilde{\Theta}_t\widetilde{H}_t\right)\Big)\right] \\ &+ V_t^{-1}\left[\gamma_t B_t + 2\Big(\gamma_t C_t R_t + \gamma_t K_t\Xi_t + \theta_t\left(\Delta_t + L_t M_t + \Theta_t H_t\right)\Big)\right]. \end{aligned} \tag{6.24}$$

6.5 Option pricing for second chaos stochastic volatility models

We now come to the option pricing problem. The model presented above is perhaps too general in terms of the number of degrees of freedom. However, this point can be addressed by requiring a number of the underlying functions to be equal. We view this generality as a positive element, one has to decide on what to calibrate, and choose the appropriate number of degrees of freedom. In this section we present option pricing in its generality; later, however, we consider a sub–model of greater tractability.

The standard vanilla call option that expires at time T and has a strike price K has a present value given by

$$H_0 = E\left[\frac{\widetilde{V}_T}{\widetilde{V}_0}\left(S_T - K\right)^+\right], \tag{6.25}$$

where now the model for the price process $\{S_t\}$ is given in (6.18). The expression inside the expectation is a function of Gaussian martingales, some of them being independent. In particular, let us first define the following random variables, all being standard normally distributed:

$$\begin{array}{llll} Z_T^1 = \dfrac{L_T}{\sqrt{N_T}}, & Z_T^2 = \dfrac{R_T}{\sqrt{Q_T}}, & Z_T^3 = \dfrac{K_t}{\sqrt{\Omega_T^k}}, & Z_T^4 = \dfrac{\Theta_T}{\sqrt{\Omega_T^\theta}}, \\ Z_T^5 = \dfrac{\widetilde{L}_T}{\sqrt{\widetilde{N}_T}}, & Z_T^6 = \dfrac{\widetilde{R}_T}{\sqrt{\widetilde{Q}_T}}, & Z_T^7 = \dfrac{\widetilde{K}_T}{\sqrt{\widetilde{\Omega}_T^k}}, & Z_T^8 = \dfrac{\widetilde{\Theta}_T}{\sqrt{\widetilde{\Omega}_T^\theta}}. \end{array} \tag{6.26}$$

We further define the following quadratic variations:

$$\Omega_T^k = \int_0^T k_s^2 ds, \qquad \Omega_T^\theta = \int_0^T \theta_s^2 ds, \qquad \widetilde{\Omega}_T^k = \int_0^T \widetilde{k}_s^2 ds, \qquad \widetilde{\Omega}_T^\theta = \int_0^T \widetilde{\theta}_s^2 ds. \tag{6.27}$$

To proceed further, we define the following polynomials:

$$\begin{aligned} \mathcal{P}_1(x) &= D_T + E_T\sqrt{N_T}x + F_T N_T(x^2-1), \\ \mathcal{P}_2(x) &= A_T + B_T\sqrt{Q_T}x + C_T Q_T(x^2-1) \\ \mathcal{P}_3(x,y) &= 2x\sqrt{\Omega_T^\theta}\left(\Delta_T + y\sqrt{N_T}M_T\right), \\ \mathcal{P}_4(x,y) &= 2x\sqrt{\Omega_T^k}\left(\Phi_T + y\sqrt{Q_T}\Xi_T\right), \end{aligned}$$

$$
\begin{aligned}
\mathcal{P}_5(x) &= H_T \Lambda_T (x^2-1) + Y_T, \\
\mathcal{P}_6(x) &= \Psi_T \Sigma_T (x^2-1) + \Gamma_T, \\
\mathcal{P}_7(x) &= \widetilde{D}_T + \widetilde{E}_T \sqrt{\widetilde{N}_T} x + \widetilde{F}_T \widetilde{N}_T (x^2-1), \\
\mathcal{P}_8(x) &= \widetilde{A}_T + \widetilde{B}_T \sqrt{\widetilde{Q}_T} x + \widetilde{C}_T \widetilde{Q}_T (x^2-1), \\
\mathcal{P}_9(x,y) &= 2\sqrt{\widetilde{\Omega}^{\theta}_T} x \left(\widetilde{\Delta}_T \sqrt{\widetilde{N}_T \widetilde{M}_T} y \right), \\
\mathcal{P}_{10}(x,y) &= 2\sqrt{\widetilde{\Omega}^{k}_T} x \left(\widetilde{\Phi}_T + y \sqrt{\widetilde{Q}_T} \widetilde{\Xi}_T \right), \\
\mathcal{P}_{11}(x) &= \widetilde{H}_T \widetilde{\Lambda}_T (x^2-1) + \widetilde{Y}_T \\
\mathcal{P}_{12}(x) &= \widetilde{\Psi}_T \widetilde{\Sigma}_T (x^2-1) + \widetilde{\Gamma}_T .
\end{aligned}
\tag{6.28}
$$

If we now consider the vector process

$$
\mathbf{Z} = [Z^1, Z^2, Z^3, Z^4, Z^5, Z^6, Z^7, Z^8], \tag{6.29}
$$

the value of the option is given by the following multiple integral:

$$
\begin{aligned}
H_0 &= \frac{1}{\widetilde{V}_0} \int_\Gamma \Big(\mathcal{P}_1(z_1) + \mathcal{P}_2(z_2) + \mathcal{P}_3(z_4, z_1) + \mathcal{P}_4(z_3, z_2) + \mathcal{P}_5(z_4) + \mathcal{P}_6(z_3) \\
&- K \left(\mathcal{P}_7(z_5) + \mathcal{P}_8(z_6) + \mathcal{P}_9(z_8, z_5) + \mathcal{P}_{10}(z_7, z_6) + \mathcal{P}_{11}(z_8) + \mathcal{P}_{12}(z_7) \right) \Big) f(\mathbf{z}) d\mathbf{z},
\end{aligned}
\tag{6.30}
$$

where the integral is taken over the region Γ for which the integral is non–negative. The function $f(\mathbf{z})$ appearing in the expression above is the standard multivariate normal density function. What we have here is an eight–dimensional integral whose solution is the option price at time zero. We are encouraged by two facts. First, it is important to mention again that the random variables involved here are normal, which makes the numerical calculations potentially tractable. The second comment is that the generality presented above can be eliminated to a degree that is left to our specification. By imposing equality conditions between some of the chaos coefficients, the inputs of our model, we can generate models that are easier to handle. We review such an example in the next section.

6.6 The reduced second chaos model

Based on results of the previous sections, we propose here a two–factor model, which we shall call a reduced second chaos model. This model belongs to the family of models considered in the previous section, and is of a great simplicity compared to the general framework. The theoretical background and terminology are the same as before. We further impose the following conditions:

$$\begin{aligned} \zeta_s = k_s = \widetilde{\zeta}_s = \widetilde{k}_s \\ \gamma_s = \theta_s = \widetilde{\gamma}_s = \widetilde{\theta}_s. \end{aligned} \tag{6.31}$$

As a result, all expressions are functions of a pair state variables that are standard normal and *independent.* More precisely, we now have:

$$\begin{aligned} Z^1 = Z^3 = Z^5 = Z^7 \equiv Z, \\ Z^2 = Z^4 = Z^6 = Z^8 \equiv \hat{Z}, \end{aligned} \tag{6.32}$$

for the two normally distributed random variables with zero correlation, and also we have:

$$\begin{aligned} N_t = \Omega_t^k = \widetilde{N}_t = \widetilde{\Omega}_t^k, \\ Q_t = \Omega_t^\theta = \widetilde{Q}_t = \widetilde{\Omega}_t^\theta, \\ \Lambda_t = \widetilde{\Lambda}_t, \quad \Sigma_t = \widetilde{\Sigma}_t. \end{aligned} \tag{6.33}$$

Calculations become straightforward under these considerations. First we define the following deterministic variables:

$$\begin{aligned} A &= D_T - K\widetilde{D}_T - N_T(F_T - K\widetilde{F}_T) - \Sigma_T(\Psi_T - K\widetilde{\Psi}_T) + \Gamma_T - K\widetilde{\Gamma}_T, \\ B &= \sqrt{N}_T(E_T - K\widetilde{E}_T) + 2\sqrt{\Omega_T^k}(\Phi_T - K\widetilde{\Phi}_T) \\ C &= N_T(F_T - K\widetilde{F}_T) + \Sigma_T(\Psi_T - K\widetilde{\Psi}_T), \end{aligned} \tag{6.34}$$

$$\begin{aligned} D &= A_T - K\widetilde{A}_T - \Lambda_T(H_T - K\widetilde{H}_T) - Q_T(C_T - K\widetilde{C}_T) + Y_T - K\widetilde{Y}_T \\ E &= \sqrt{Q}_T(B_T - K\widetilde{B}_T) + 2\sqrt{\Omega_T^\theta}(\Delta_T - K\widetilde{\Delta}_T) \\ F &= Q_T(C_T - K\widetilde{C}_T) + \Lambda_T(H_T - K\widetilde{H}_T). \end{aligned} \tag{6.35}$$

Further we define the following function:

$$X = 2\sqrt{\Omega_T^k}\Big(\sqrt{Q_T}(\Xi_T - K\widetilde{\Xi}_T) + \sqrt{N_T}(M_T - K\widetilde{M}_T)\Big). \tag{6.36}$$

The definitions above lead to a much simpler representation of the option price within this model. If we make use of the polynomials

$$P_1(x) = A + Bx + Cx^2, \qquad P_2(x) = D + Ex + Fx^2, \tag{6.37}$$

then we obtain the following expression:

$$H_0 = \frac{1}{\widetilde{V}_0} E\left[\left(P_1(Z) + P_2(\hat{Z}) + Z\hat{Z}X\right)^+\right], \tag{6.38}$$

which can be written in the form

$$H_0 = \frac{1}{\widetilde{V}_0}\int_{-\infty}^{\infty}\int_{z^*}^{\infty} \left(P_1(z) + P_2(\hat{z}) + z\hat{z}X\right)\rho(z,\hat{z})dzd\hat{z}, \tag{6.39}$$

where now we write $\rho(z,\hat{z})$ for the standard bivariate normal density function with zero correlation.

6.7 Concluding remarks

One wonders what makes a good model of option pricing. The model should do this exactly: price options. But to the mathematician this is not enough: the assumptions, sophistication, and flexibility, all based on the standard theory of finance developed over the last thirty–five years, are important as well. We have proposed here a model that makes use of the standard theories within a new framework. Essentially the 'pricing kernel' approach we consider from the beginning of this thesis, separates the issues of pricing and hedging derivatives. To be sure, when we know how to hedge, we know how to price. When a contingent claim can be replicated perfectly by use of other underlying assets, then its price is uniquely determined. However, this state of affairs does not work when a perfect hedge cannot be found. Still, when there is no hedge, this does not necessarily mean the pricing game has been lost, and this is where the pricing kernel method comes into play. Within the chaotic framework, by introducing a pricing kernel process $\{V_t\}$, through the choice of a specific chaos expansion, we determine unique option prices, even within an incomplete market model. The determination of the pricing kernel, on the other hand, can be viewed as connected to the aggregate risk preferences of market agents (investors, traders, speculators, etc.). From a practical point of view, we observe that a liquid product will always have a unique price, the one we see in the market. The hedging issue is different in many respects, and one can, and no doubt should, think of a perfect hedge as a lost game in many (incomplete market) cases. Still, one may be able to price. The relative pricing of derivatives is at the core of standard finance theory. That is to say, derivatives are priced relative to the underlying assets on which they are written. A rather different point of view, would be to think of derivatives as 'underlying' assets themselves, priced in the real world, under the 'real' probability measure.

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